

The fundamental group as a topological group

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Abstract

This paper is devoted to the study of a natural group topology on the fundamental group which remembers local properties of spaces forgotten by covering space theory and weak homotopy type. It is known that viewing the fundamental group as the quotient of the loop space often fails to result in a topological group; we use free topological groups to construct a topology which promotes the fundamental group of any space to topological group structure. The resulting invariant, denoted π_1^t , takes values in the category of topological groups, can distinguish spaces with isomorphic fundamental groups, and agrees with the quotient fundamental group precisely when the quotient topology yields a topological group. Most importantly, this choice of topology allows us to naturally realize free topological groups and pushouts of topological groups as fundamental groups via topological analogues of classical results in algebraic topology.

1 Introduction

Classical results in basic algebraic topology give that groups are naturally realized as fundamental groups of spaces. For instance, free groups arise as fundamental groups of wedges of circles, any group can be realized as the fundamental group of some CW-complex, and pushouts of groups arise via the van Kampen theorem. This ability to construct geometric interpretations of discrete groups has important applications in both topology and algebra. It is natural to ask if this symbiotic relationship can be extended, via some topological version of the invariant, so that one can study topological groups by studying spaces with homotopy type other than that of a CW-complex and vice versa.

This paper is devoted to one such topologically enriched version of the fundamental group. In particular, the fundamental group is endowed with a group topology which can be used to study spaces with complicated local structure. To approach such spaces by way of directly transferring topological data to a homotopy invariant is in contrast with the more historical shape theoretic approach where spaces are approximated by polyhedra and pro-groups take the place of groups. Our view is quickly justified, however, as we find natural connections between locally complicated spaces and the general theory of topological groups not possible in shape theory. For instance, the use of (non-discrete) topological groups in the place of groups allow us to give “realization” results (See Section 4) in the form of topological analogues of the classical results mentioned above. Additionally, we set the foundation for a theory of generalized covering maps (called semicoverings) to appear in [6].

In making a choice of topology on π_1 , it is inevitable that we ignore many interesting topologies likely to have their own benefits and uses. The topology introduced in the present paper is partially motivated by the quotient topology used first by Biss [3]. Specifically, $\pi_1(X, x_0)$ may be viewed as the quotient space of the loop space $\Omega(X, x_0)$ (with the compact-open topology) with respect to the natural function $\Omega(X, x_0) \rightarrow \pi_1(X, x_0)$ identifying path components. This construction results in a functor $\pi_1^{qtop} : \mathbf{hTop}_* \rightarrow \mathbf{qTopGrp}$ from the homotopy category of based spaces to the category of quasitopological groups¹ and continuous homomorphisms [3, 7], however, recent results indicate that very often $\pi_1^{qtop}(X, x_0)$ fails to be a topological group. In

¹A quasitopological group is a group with topology such that inversion is continuous and multiplication is continuous in each variable. See [1] for basic theory.

fact, this failure occurs even for one-dimensional planar, Peano continua [13] and locally simply-connected (but non-locally path connected) planar sets [4, 14]. For this reason, we refer to π_1^{qtop} as the *quasitopological fundamental group*.

When considering which properties a “topological” fundamental group should have, the continuity of the natural function $\Omega(X, x_0) \rightarrow \pi_1(X, x_0)$ is certainly among the most useful. Indeed, a topology on $\pi_1(X, x_0)$ should remember something about the topology of loops representing homotopy classes. By definition, the quotient topology of $\pi_1^{qtop}(X, x_0)$ is the finest topology on $\pi_1(X, x_0)$ with this property. On the other hand, $\pi_1^{qtop}(X, x_0)$ is not always a topological group. One might then ask if there is a finest *group topology* on $\pi_1(X, x_0)$ such that $\Omega(X, x_0) \rightarrow \pi_1(X, x_0)$ is continuous. We find that the existence of this topology follows directly from the existence of the free topological groups in the sense of Markov. The resulting topological group, denoted $\pi_1^\tau(X, x_0)$, is invariant under homotopy equivalence and retains information beyond the covering space theory of X . The construction and basic theory of $\pi_1^\tau(X, x_0)$ appears in Section 3 after notation and preliminary facts are discussed in Section 2.

The appearance of free topological groups in [4] also motivates the construction of $\pi_1^\tau(X, x_0)$. There is a vast literature of free topological groups and free topological products, however, explicit descriptions of these groups are often quite complicated. While it is unlikely such groups arise naturally in shape theory, we find they appear with great generality in topological analogues of classical computational results involving π_1^τ . For instance, just as the fundamental group of a wedge of circles $\bigvee_X S^1$ is free on a discrete set X , π_1^τ evaluated on a “generalized wedge of circles” $\Sigma(X_+)$ on arbitrary X is free topological on the path component space of X (Theorem 4.1). Secondly, mimicking the usual proof that every group is realized as a fundamental group, it is possible to realize every topological group as the fundamental group $\pi_1^\tau(Y, y_0)$ of a space Y obtained by attaching 2-cells to a generalized wedge (Theorem 4.10). Finally, a topological van Kampen theorem (Theorem 4.23) enhances the computability of π_1^τ in terms of pushouts of topological groups. These three results compose the three parts of Section 4.

2 Preliminaries and notation

Prior to constructing $\pi_1^\tau(X, x_0)$, we recall a few basic constructions and facts. For spaces X, Y , let $M(X, Y)$ denote the set of continuous maps $X \rightarrow Y$ with the compact open topology generated by subbasis sets $\langle K, U \rangle = \{f \mid f(K) \subset U\}$ where $K \subseteq X$ is compact and U is open in Y . If $A \subseteq X$ and $B \subseteq Y$, let $M((X, A), (Y, B))$ denote the subspace of maps such that $f(A) \subseteq B$. If $A = \{x\}$ and $B = \{y\}$ are basepoints of X and Y , we write $M_*(X, Y)$. If $f : Y \rightarrow Z$ is a map, let $f_\# : M(X, Y) \rightarrow M(X, Z)$, $k \mapsto f \circ k$ be the continuous map induced on mapping spaces. In particular, let $P(X)$ be the free path space $M(I, X)$ where $I = [0, 1]$ is the unit interval. Clearly this construction results in a functor $P : \mathbf{Top} \rightarrow \mathbf{Top}$, where $P(f) = f_\#$ on morphisms.

Suppose X is a space and \mathcal{B}_X is a basis for the topology of X which is closed under finite intersection (for instance, the topology of X itself). We are interested in using a convenient basis for the compact-open topology of $P(X)$. This basis consists of neighborhoods of the form $\bigcap_{j=1}^n \langle K_n^j, U_j \rangle$ where $K_n^j = \left[\frac{j-1}{n}, \frac{j}{n} \right]$ and $U_j \in \mathcal{B}_X$. We frequently use these neighborhoods since they are easy to manipulate and allow us to intuit basic open neighborhoods in $P(X)$ as finite, ordered sets of “instructions.”

The following notation for subspaces of $P(X)$ will be used: For $x_0, x_1 \in X$, let $P(X, x_0) = \{\alpha \in P(X) \mid \alpha(0) = x_0\}$, $P(X, x_0, x_1) = \{\alpha \in P(X) \mid \alpha(i) = x_i, i = 0, 1\}$, and $\Omega(X, x_0) = P(X, x_0, x_0)$. When the choice of basepoint is understood, we often write $\Omega(X)$ for the loop space. In notation, we will not always distinguish a neighborhood $\bigcap_{j=1}^n \langle C_j, U_j \rangle$ from being an open set in $P(X)$ or any of its subspaces. We say a loop $\alpha \in \Omega(X, x)$ is *trivial* if it is homotopic to the constant loop c_x at x and *non-trivial* if it is not trivial.

We make use of the following notation for restricted paths and neighborhoods as in [4]. For any fixed, closed subinterval $A \subseteq I$, let $H_A : I \rightarrow A$ be the unique, increasing, linear homeomorphism. For a path $p : I \rightarrow X$, the *restricted path of p to A* is the composite $p_A = p|_A \circ H_A : I \rightarrow A \rightarrow X$. As a convention, if $A = \{t\} \subseteq I$ is a singleton, p_A will denote the constant path at $p(t)$. Note that if $0 = t_0 \leq t_1 \leq \dots \leq t_n = 1$, knowing the paths $p_{[t_{i-1}, t_i]}$ for $i = 1, \dots, n$ uniquely determines p . It is simple to describe concatenations of paths with this notation: If $p_1, \dots, p_n : I \rightarrow X$ are paths such that $p_j(1) = p_{j+1}(0)$ for each $j = 1, \dots, n-1$, then the *n -fold concatenation*

of this sequence of paths is the unique path $q = p_1 * p_2 * \dots * p_n$ such that $q_{K_n^i} = p_j$ for each $j = 1, \dots, n$. It is a basic fact of the compact-open topology that concatenation $P(X) \times_X P(X) = \{(\alpha, \beta) | \alpha(1) = \beta(0)\} \rightarrow P(X)$, $(\alpha, \beta) \mapsto \alpha * \beta$ is continuous. If $\alpha \in P(X)$, then $\alpha^{-1}(t) = \alpha(1 - t)$ is the *reverse* of α and for a set $A \subseteq P(X)$, $A^{-1} = \{\alpha^{-1} | \alpha \in A\}$. The operation $\alpha \mapsto \alpha^{-1}$ is a self-homeomorphism of $P(X)$.

Let $\mathcal{U} = \bigcap_{j=1}^n \langle C_j, U_j \rangle$ be an open neighborhood of a path $p \in P(X)$. Then $\mathcal{U}_A = \bigcap_{A \cap C_j \neq \emptyset} \langle H_A^{-1}(A \cap C_j), U_j \rangle$ is an open neighborhood of p_A called the *restricted neighborhood* of \mathcal{U} to A . If $A = \{t\}$ is a singleton, then $\mathcal{U}_A = \bigcap_{t \in C_j} \langle I, U_j \rangle = \langle I, \bigcap_{t \in C_j} U_j \rangle$. On the other hand, if $p = q_A$ for some path $q \in P(X)$, then $\mathcal{U}^A = \bigcap_{j=1}^n \langle H_A(C_j), U_j \rangle$ is an open neighborhood of q called the *induced neighborhood* of \mathcal{U} on A . If A is a singleton so that p_A is a constant map, let $\mathcal{U}^A = \bigcap_{j=1}^n \langle \{t\}, U_j \rangle$. We frequently make use of the following Lemma which is straightforward to verify.

Lemma 2.1. *Let $\mathcal{U} = \bigcap_{j=1}^n \langle C_j, U_j \rangle$ be an open neighborhood in $P(X)$ such that $\bigcup_{j=1}^n C_j = I$. Then*

1. *For any closed interval $A \subseteq I$, $(\mathcal{U}^A)_A = \mathcal{U} \subseteq (\mathcal{U}_A)^A$*
2. *If $0 = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_n = 1$, then $\mathcal{U} = \bigcap_{i=1}^n (\mathcal{U}_{[t_{i-1}, t_i]})^{[t_{i-1}, t_i]}$.*

3 A group topology on the fundamental group

3.1 Free topological groups and the reflection τ

The *free (Markov) topological group* on an unbased space Y is the unique topological group $F_M(Y)$ with a continuous map $\sigma : Y \rightarrow F_M(Y)$ universal in the sense that for any map $f : Y \rightarrow G$ to a topological group G , there is a unique continuous homomorphism $\tilde{f} : F_M(Y) \rightarrow G$ such that $f = \tilde{f} \circ \sigma$. One can show that $F_M(Y)$ exists by showing the forgetful functor $\mathbf{TopGrp} \rightarrow \mathbf{Top}$ from the category of topological groups to topological spaces has a left adjoint $F_M : \mathbf{Top} \rightarrow \mathbf{TopGrp}$. This is achieved by an application of Taut liftings or the General Adjoint Functor Theorem [32]. The underlying group of $F_M(Y)$ is simply the free group $F(Y)$ on the underlying set of Y and $\sigma : Y \rightarrow F_M(Y)$ is the canonical injection of generators. The reader is referred to [34] for proofs of the following basic facts.

Lemma 3.1. *Let X and Y be topological spaces.*

1. *$F_M(Y)$ is Hausdorff (discrete) if and only if Y is functionally Hausdorff² (discrete).*
2. *The canonical map $\sigma : Y \rightarrow F_M(Y)$ is an embedding if and only if Y is completely regular.*
3. *If $q : X \rightarrow Y$ is a quotient map, then so is $F_M(q) : F_M(X) \rightarrow F_M(Y)$.*

We use free topological groups to make the following useful construction:

A *group with topology* is a group G with a topology but where no restrictions are made on the continuity of the operations. The topology of G will typically be denoted \mathcal{T}_G . Let $\mathbf{GrpwTop}$ be the category of groups with topology and continuous homomorphisms. Given any $G \in \mathbf{GrpwTop}$, the identity $G \rightarrow G$ induces the multiplication epimorphism $m_G : F(G) \rightarrow G$. Give G the quotient topology with respect to $m_G : F(G) \rightarrow G$ and denote the resulting group as $\tau(G)$.

In general, if $p : H \rightarrow G$ is an epimorphism of groups and H is a topological group, G becomes a topological group when it is given the quotient topology with respect to p . Therefore, $\tau(G)$ is a topological group. The identity $G \rightarrow \tau(G)$ is continuous since it is the composition $m_G \circ \sigma : G \rightarrow F_M(G) \rightarrow \tau(G)$ and, moreover, $\tau(G)$ enjoys the universal property: *If $f : G \rightarrow H$ is any continuous homomorphism to a topological group H , then $f : \tau(G) \rightarrow H$ is continuous.*

²A space is functionally Hausdorff if distinct points may be separated by continuous real valued functions.

Indeed, f induces a continuous homomorphism $\tilde{f} : F_M(G) \rightarrow H$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\sigma} & F_M(G) \\ & \searrow id & \downarrow m_G \\ & & \tau(G) \end{array} \quad \begin{array}{ccc} & & \searrow \tilde{f} \\ & & \downarrow \\ & & H \end{array} \quad \begin{array}{ccc} & & \xrightarrow{f} \end{array}$$

commutes. Since m_G is quotient, $f : \tau(G) \rightarrow H$ is continuous.

Stated entirely in categorical terms this amounts to the fact that **TopGrp** is a full reflective subcategory of **GrpwTop**.

Lemma 3.2. $\tau : \mathbf{GrpwTop} \rightarrow \mathbf{TopGrp}$ is a functor left adjoint to the inclusion functor $i : \mathbf{TopGrp} \hookrightarrow \mathbf{GrpwTop}$. Moreover, each reflection map $r_G : G \rightarrow \tau(G)$ is the continuous identity homomorphism.

Proof. Define τ to be the identity on underlying groups and homomorphisms. Thus to see that τ is a well-defined functor, it suffices to check that $\tau(f) : \tau(G) \rightarrow \tau(H)$ is continuous. Note that $F_M(f) : F_M(G) \rightarrow F_M(H)$ is a continuous homomorphism and the square

$$\begin{array}{ccc} F_M(G) & \xrightarrow{F_M(f)} & F_M(H) \\ m_G \downarrow & & \downarrow m_H \\ \tau(G) & \xrightarrow{\tau(f)} & \tau(H) \end{array}$$

of continuous homomorphisms commutes. Continuity of $\tau(f)$ follows from the fact that the left vertical map in this diagram is quotient.

The natural bijection characterizing the adjunction is $\mathbf{TopGrp}(\tau(G), H) \cong \mathbf{GrpwTop}(G, i(H))$, $f \mapsto f \circ r_G$ and follows from the universal property of $\tau(G)$ illustrated above. \square

It is often convenient to think of τ as a functor which removes the smallest number of open sets from the topology of G so that a topological group is obtained.

Proposition 3.3. *The functor τ has the following properties.*

1. τ preserves finite products.
2. τ preserves quotient maps.
3. If $G \in \mathbf{GrpwTop}$, then G is a topological group if and only if $G = \tau(G)$.
4. If $G \in \mathbf{GrpwTop}$, then G is discrete if and only if $\tau(G)$ is discrete.

Proof.

1. To see that τ preserves finite products, let $G, H \in \mathbf{GrpwTop}$. Clearly $G \times H$ with the product topology is the categorical product in **GrpwTop**. Apply τ to the projections of $G \times H$ to induce the continuous identity homomorphism $\tau(G \times H) \rightarrow \tau(G) \times \tau(H)$. The inclusions $i : G \rightarrow G \times H$, $i(g) = (g, e_H)$ and $j : H \rightarrow G \times H$, $j(h) = (e_G, h)$ are embeddings of groups with topology. Let μ be the continuous multiplication of $\tau(G \times H)$. The composition

$$\mu \circ (\tau(i) \times \tau(j)) : \tau(G) \times \tau(H) \rightarrow \tau(G \times H) \times \tau(G \times H) \rightarrow \tau(G \times H)$$

is the continuous identity proving that $id : \tau(G \times H) \cong \tau(G) \times \tau(H)$.

2. Suppose $f : G \rightarrow H$ is a homomorphism which is also a topological quotient map. Since F_M preserves quotients, $F_M(f)$ is a quotient map. The top and vertical maps in the diagram of the proof of Lemma 3.2 are all quotient. It follows that $\tau(f)$ is quotient.
3. One direction is obvious. If G is a topological group, then the identity $id : G \rightarrow G$ induces the continuous identity $\tau(G) \rightarrow G$ which is the inverse of $r_G : G \rightarrow \tau(G)$.
4. Since the identity $G \rightarrow \tau(G)$ is continuous, G is discrete whenever $\tau(G)$ is. Conversely, if G is discrete, then so is $F_M(G)$ and its quotient $\tau(G)$.

□

So far, we have only constructed $\tau(G)$ as the quotient of a free topological group. Explicit descriptions of free topological groups are known [33] but are, in general, quite complicated. For this reason, we provide an alternative approach to the topology of $\tau(G)$ when G is a quasitopological group. We follow the well-known procedure of approximating group topologies through a transfinite process of taking quotient topologies [9, 27].

If G is a quasitopological group, let $c(G)$ be the underlying group of G with the quotient topology with respect to multiplication $\mu_G : G \times G \rightarrow G$.

Proposition 3.4. $c : \mathbf{qTopGrp} \rightarrow \mathbf{qTopGrp}$ is a functor when defined to be the identity on morphisms.

Proof. For $G \in \mathbf{qTopGrp}$, consider the diagram

$$\begin{array}{ccc} G \times G & \longrightarrow & G \times G \\ \mu_G \downarrow & & \downarrow \mu_G \\ c(G) & \longrightarrow & c(G) \end{array}$$

Let $g \in c(G)$ and the top map be $(a, b) \mapsto (b^{-1}, a^{-1})$ (resp. $(a, b) \mapsto (ga, b)$, $(a, b) \mapsto (a, bg)$). Each of these functions are continuous since G is a quasitopological group. The diagram commutes when the bottom map is inversion (resp. left multiplication by g , right multiplication by g). Since the vertical maps are quotient, the universal property of quotient spaces implies that these operations in $c(G)$ are continuous. A similar argument gives that $c(f) = f : c(G) \rightarrow c(G')$ is continuous for each continuous homomorphism $f : G \rightarrow G'$ of quasitopological groups. □

Proposition 3.5. Let G be a quasitopological group.

1. The identity homomorphisms $G \rightarrow c(G) \rightarrow \tau(G)$ are continuous.
2. Then $\tau(c(G)) = \tau(G)$.
3. Then G is a topological group if and only if $G = c(G)$.

Proof. 1. Let e be the identity of G . Consider the diagram

$$\begin{array}{ccccc} G \times \{e\} & \xrightarrow{id \times e} & G \times G & \xrightarrow{r_G \times r_G} & \tau(G) \times \tau(G) \\ \mu_G \downarrow \cong & & \mu_G \downarrow & & \mu_G \downarrow \\ G & \xrightarrow{id} & c(G) & \xrightarrow{id} & \tau(G) \end{array}$$

Each vertical map is quotient and the maps in the top row are continuous. The identities in the bottom row are continuous by the universal property of quotient spaces.

2. Applying τ to $id : G \rightarrow c(G)$ gives $id : \tau(G) \rightarrow \tau(c(G))$. The adjoint of $c(G) \rightarrow \tau(G)$ is the inverse $\tau(c(G)) \rightarrow \tau(G)$. This gives the equality $\tau(c(G)) = \tau(G)$.

3. If G is a topological group, then $G = \tau(G)$ and all three topologies on G agree by the first statement. Conversely, if $G = c(G) \in \mathbf{qTopGrp}$, then $\mu_G : G \times G \rightarrow c(G) = G$ is continuous and G is a topological group. \square

The proof of the next proposition is a straightforward exercise left to the reader.

Proposition 3.6. *If G_λ is a family of quasitopological groups each with underlying group G and topology \mathcal{T}_{G_λ} , then the topology $\bigcap_\lambda \mathcal{T}_{G_\lambda}$ on G makes G a quasitopological group.*

Approximation of $\tau(G)$ 3.7. Let $G = G_0$ be a quasitopological group with topology \mathcal{T}_{G_0} . We iterate the above construction by letting $G_\alpha = c(G_{\alpha-1})$ (with topology \mathcal{T}_{G_α}) for each ordinal α with a predecessor. When α is a limit ordinal, let G_α have topology $\mathcal{T}_{G_\alpha} = \bigcap_{\beta < \alpha} \mathcal{T}_{G_\beta}$. Applying Propositions 3.4 and 3.6 in a simple transfinite induction argument gives that G_α is a quasitopological group for each ordinal α . Using a standard cardinality argument, we now show the topologies \mathcal{T}_{G_α} stabilize to the topology $\mathcal{T}_{\tau(G)}$ of $\tau(G)$.

Theorem 3.8. *There is an ordinal α such that $G_\alpha = \tau(G)$.*

Proof. We have already noted that $\mathcal{T}_{\tau(G)} \subseteq \mathcal{T}_{G_{\alpha+1}} \subseteq \mathcal{T}_{G_\alpha} \subseteq \mathcal{T}_{G_0}$ for every α . Since the identity homomorphisms $G \rightarrow G_\alpha \rightarrow \tau(G)$ are continuous, $G_\alpha = \tau(G)$ whenever G_α is a topological group. Therefore, if $G_\alpha \neq \tau(G)$, then G_α is not a topological group and Prop. 3.5 implies that $\mathcal{T}_{G_{\alpha+1}}$ is a proper subset of \mathcal{T}_{G_α} . Consequently, if $G_\alpha \neq \tau(G)$ for every ordinal α , there are distinct sets $U_\alpha \in \mathcal{T}_{G_\alpha} - \mathcal{T}_{G_{\alpha+1}} \subseteq \mathcal{T}_{G_0}$ indexed by the ordinals. This contradicts the fact that there is an ordinal which does not inject into the power set of \mathcal{T}_{G_0} . \square

Since the identity homomorphisms $G_\beta \rightarrow G_\alpha \rightarrow \tau(G)$ are continuous whenever $\beta \leq \alpha$, the groups G_α stabilize to $\tau(G)$.

Corollary 3.9. *If $G \in \mathbf{qTopGrp}$, then G and $\tau(G)$ have the same open subgroups.*

Proof. Since the identity $G \rightarrow \tau(G)$ is continuous, every open subgroup of $\tau(G)$ is open in G . For the converse, we check that every open subgroup of G is open in $c(G)$ and proceed by transfinite induction.

Suppose H is an open subgroup of G . Note that if $\mu_G : G \times G \rightarrow G$ is multiplication, then $\mu_G^{-1}(H) = \bigcup_{ab \in H} Hb^{-1} \times a^{-1}H$. Since G is a quasitopological group and H is open in G , $\mu_G^{-1}(H)$ is open in $G \times G$. By construction, $\mu_G : G \times G \rightarrow c(G)$ is quotient and therefore H is open in $c(G)$.

Suppose H is an open subgroup of $G = G_0$. Thus $H \in \mathcal{T}_{G_0}$. If $H \in \mathcal{T}_{G_\beta}$ for each $\beta < \alpha$, then certainly $H \in \mathcal{T}_{G_\alpha} = \bigcap_{\beta < \alpha} \mathcal{T}_{G_\beta}$ if α is a limit ordinal. If α is a successor ordinal, then $H \in \mathcal{T}_{G_{\alpha-1}}$ and the previous paragraph gives that $H \in \mathcal{T}_{c(G_{\alpha-1})} = \mathcal{T}_{G_\alpha}$. Therefore H is open in G_α for every ordinal α . Since the G_α stabilize to $\tau(G)$, H is open in $\tau(G)$. \square

The category **TopGrp** is complete and cocomplete and the underlying group of the colimit/limit of a diagram $J \rightarrow \mathbf{TopGrp}$ agrees with the limit/colimit of the diagram $J \rightarrow \mathbf{TopGrp} \rightarrow \mathbf{Grp}$ of underlying groups. While limits in **TopGrp** have obvious descriptions, the topology of a colimit can (as free topological groups) be quite complicated. The topological van Kampen theorem presented in Section 4.3 motivates a brief note on the relationship between τ and pushouts in **TopGrp**. Let $G_1 *_G G_2$ denote the pushout (or *free topological product with amalgamation*) of a diagram $G_1 \leftarrow G \rightarrow G_2$. If $G = \{*\}$, this is simply the *free topological product* $G_1 * G_2$. The topology of $G_1 * G_2$ is quotient with respect to the projection $k_{G_1, G_2} : F_M(G_1 \sqcup G_2) \rightarrow G_1 * G_2$ (where $G_1 \sqcup G_2$ is the coproduct in **Top**) and $G_1 *_G G_2$ is quotient with respect to the projection $G_1 * G_2 \rightarrow G_1 *_G G_2$.

Proposition 3.10. *For groups with topology G_1, G_2 , the canonical epimorphism $k_{G_1, G_2} : F_M(G_1 \sqcup G_2) \rightarrow \tau(G_1) * \tau(G_2)$ is a topological quotient map.*

Proof. The following diagram commutes in the category of topological groups.

$$\begin{array}{ccc}
 F_M(F_M(G_1) \sqcup F_M(G_2)) & \xrightarrow{k_{F_M(G_1), F_M(G_2)}} & F_M(G_1) * F_M(G_2) \xrightarrow{\cong} F_M(G_1 \sqcup G_2) \\
 \downarrow F_M(m_{G_1 \sqcup G_2}) & & \downarrow k_{G_1, G_2} \\
 F_M(\tau(G_1) \sqcup \tau(G_2)) & \xrightarrow{k_{\tau(G_1), \tau(G_2)}} & \tau(G_1) * \tau(G_2)
 \end{array}$$

Since m_{G_1}, m_{G_2} are quotient and F_M preserves quotients, all maps except for the right vertical map are known to be quotient. By the universal property of quotient spaces, $k_{G_1, G_2} : F_M(G_1 \sqcup G_2) \rightarrow \tau(G_1) * \tau(G_2)$ must also be quotient. \square

3.2 The construction and characterizations of $\pi_1^\tau(X, x_0)$

The *path component space* of a space X , denoted $\pi_0^{qtop}(X)$, is the set of path components $\pi_0(X)$ with the quotient topology with respect to the canonical function $X \rightarrow \pi_0(X)$ identifying path components. Since a map $X \rightarrow Y$ induces a continuous map $\pi_0^{qtop}(X) \rightarrow \pi_0^{qtop}(Y)$ on path component spaces, we obtain a functor $\pi_0^{qtop} : \mathbf{Top} \rightarrow \mathbf{Top}$.

The path component space $\pi_1^{qtop}(X, x_0) = \pi_0^{qtop}(\Omega(X, x_0))$ is the *quasitopological fundamental group* of (X, x_0) . Since multiplication and inversion in the fundamental group are induced by the continuous operations $(\alpha, \beta) \mapsto \alpha * \beta$ and $\alpha \mapsto \alpha^{-1}$ in the loop space, it is immediate from the universal property of quotient spaces that $\pi_1^{qtop}(X, x_0)$ is a quasitopological group.

Since a based map $f : (X, x_0) \rightarrow (Y, y_0)$ induces a continuous group homomorphism $f_* = \pi_0^{qtop}(\Omega(f)) : \pi_1^{qtop}(X, x_0) \rightarrow \pi_1^{qtop}(Y, y_0)$, this construction results in a functor $\pi_1^{qtop} : \mathbf{hTop}_* \rightarrow \mathbf{qTopGrp}$ on the homotopy category of based spaces. It is worth noting that the isomorphism class of the quasitopological fundamental group is independent of the choice of basepoint (See the proof of Proposition 3.15 below). We write $\pi_1^{qtop}(X)$ when the choice of basepoint is understood.

Construction of π_1^τ 3.11. As mentioned in the introduction, it is known that $\pi_1^{qtop}(X, x_0)$ fails to be a topological group even for compact planar sets, however, since $\mathbf{qTopGrp}$ is a full subcategory of $\mathbf{GrpwTop}$, the composition of functors $\pi_1^\tau = \tau \circ \pi_1^{qtop} : \mathbf{hTop}_* \rightarrow \mathbf{TopGrp}$ is well-defined. This new functor assigns, to a based space X , a topological group $\pi_1^\tau(X)$ whose underlying group is $\pi_1(X)$. Since the identity homomorphism $\pi_1^{qtop}(X) \rightarrow \pi_1^\tau(X)$ is continuous, so is

$$\pi : \Omega(X) \rightarrow \pi_1^{qtop}(X) \rightarrow \pi_1^\tau(X).$$

The topological group $\pi_1^\tau(X)$ can be characterized in a number of ways: According to the explicit construction of τ , $\pi_1^\tau(X)$ is the quotient of the free topological group $F_M(\pi_1^{qtop}(X))$. Additionally, since F_M preserves quotient maps, $F_M(\pi) : F_M(\Omega(X)) \rightarrow F_M(\pi_1^{qtop}(X))$ is quotient and thus $\pi_1^\tau(X)$ is the quotient of $F_M(\Omega(X))$ with respect to the map $\alpha_1 \dots \alpha_n \mapsto [\alpha_1 * \dots * \alpha_n]$. It is also convenient to characterize $\pi_1^\tau(X)$ in terms of its universal property.

Proposition 3.12. *If $\Phi : \pi_1^\tau(X) \rightarrow G$ is a homomorphism to a topological group G such that $\Phi \circ \pi : \Omega(X) \rightarrow \pi_1^\tau(X) \rightarrow G$ is continuous, then Φ is also continuous.*

Proof. If $\Phi \circ \pi : \Omega(X) \rightarrow G$ is continuous, then $\Phi : \pi_1^{qtop}(X) \rightarrow G$ is continuous by the universal property of quotient spaces. Since G is a topological group, the adjoint $\Phi : \pi_1^\tau(X) \rightarrow G$ is continuous. \square

Theorem 3.13. *The topology of $\pi_1^\tau(X)$ is the finest group topology on $\pi_1(X)$ such that $\pi : \Omega(X) \rightarrow \pi_1(X)$ is continuous.*

Proof. Suppose G is a topological group with underlying group $\pi_1(X)$ and is such that $\pi : \Omega(X) \rightarrow G$ is continuous. Since $id \circ \pi : \Omega(X) \rightarrow \pi_1^\tau(X) \rightarrow G$ is continuous, $id : \pi_1^\tau(X) \rightarrow G$ is continuous by Proposition 3.12. Thus the topology of $\pi_1^\tau(X)$ is finer than that of G . \square

Remark 3.14. Fabel [14] has recently shown that for each $n \geq 2$ the quasitopological homotopy group $\pi_n^{qtop} = \pi_1^{qtop} \circ \Omega^{n-1}$, first studied in [20, 21, 22], fails to take values in the category of topological abelian groups. We can then define $\pi_n^\tau = \tau \circ \pi_n^{qtop} = \pi_1^\tau \circ \Omega^{n-1}$ to obtain a truly “topological” higher homotopy group. In the present paper, we do not give any special attention to these higher homotopy groups.

3.3 Basic properties of $\pi_1^\tau(X, x_0)$

Since π_1^τ is defined as the composition $\tau \circ \pi_1^{qtop}$, it is often practical to approach the topology of $\pi_1^\tau(X)$ by studying the quotient topology and the behavior of τ separately. For instance, if $h : \pi_1^{qtop}(Y) \cong \pi_1^{qtop}(X)$ as quasitopological groups, then $\tau(h) : \pi_1^\tau(X) \cong \pi_1^\tau(Y)$ as topological groups. In this way, the homotopy invariance of π_1^τ follows from the homotopy invariance of π_1^{qtop} .

The following proposition indicates the isomorphism class of $\pi_1^\tau(X)$ (for path connected X) is independent of the choice of basepoint.

Proposition 3.15. *If $\gamma : I \rightarrow X$ is a path, then $\pi_1^\tau(X, \gamma(1)) \rightarrow \pi_1^\tau(X, \gamma(0))$, $[\alpha] \mapsto [\gamma * \alpha * \gamma^{-1}]$ is an isomorphism of topological groups. Consequently, if x_0, x_1 lie in the same path component of X , then $\pi_1^\tau(X, x_0) \cong \pi_1^\tau(X, x_1)$.*

Proof. The continuous operation $c_\gamma : \Omega(X, \gamma(1)) \rightarrow \Omega(X, \gamma(0))$ given by $\alpha \mapsto \gamma * \alpha * \gamma^{-1}$ induces the continuous, group isomorphism $\Gamma : \pi_1^{qtop}(X, \gamma(1)) \rightarrow \pi_1^{qtop}(X, \gamma(0))$, $[\alpha] \mapsto [\gamma * \alpha * \gamma^{-1}]$ on path component spaces. The inverse is continuous since $c_{\gamma^{-1}}$ is continuous. Thus Γ is an isomorphism of quasitopological groups and $\tau(\Gamma)$ is an isomorphism of topological groups. \square

Recall from 3. of Proposition 3.3 that $\pi_1^\tau(X) = \pi_1^{qtop}(X)$ if and only if $\pi_1^{qtop}(X)$ is a topological group. Thus if $\pi_1^{qtop}(X)$ fails to be a topological group, the topology of $\pi_1^\tau(X)$ is strictly coarser than that of $\pi_1^{qtop}(X)$. Despite this fact that some open sets may “be removed” from the quotient topology on $\pi_1(X)$ by applying τ , these two groups always share the same open subgroups.

Proposition 3.16. *For any based space X , $\pi_1^{qtop}(X)$ and $\pi_1^\tau(X)$ have the same open subgroups.*

Proof. This is a special case of Corollary 3.9. \square

The following characterization of X for which $\pi_1^\tau(X)$ is a discrete group reinforce the idea that the topology of $\pi_1^\tau(X)$ is defined to remember only non-trivial, local homotopical properties of spaces.

Proposition 3.17. *For any path connected space X , the following are equivalent:*

1. $\pi_1^\tau(X)$ is a discrete group.
2. $\pi_1^{qtop}(X)$ is a discrete group.
3. For every null-homotopic loop $\alpha \in \Omega(X)$, there is an open neighborhood \mathcal{U} of α in $\Omega(X)$ containing only null-homotopic loops.

Proof. 1. \Leftrightarrow 2. is a special case of the second part of Proposition 3.3. 2. \Leftrightarrow 3. follows from the definition of the quotient topology. \square

It is more convenient to characterize discreteness in terms of local properties of X itself by applying the characterization of discreteness of the quasitopological fundamental group in [7].

Corollary 3.18. *Suppose X is path connected. If $\pi_1^\tau(X)$ is discrete, then X is semilocally 1-connected. If X is locally path connected and semilocally 1-connected, then $\pi_1^\tau(X)$ is discrete.*

Thus $\pi_1^\tau(X)$ is discrete when X has the homotopy type of a CW-complex, manifold, or, more generally, any locally path connected space with a universal covering space.

Since $\pi_1^{qtop}(X)$ is not always a topological group, π_1^{qtop} does not preserve finite products. The following proposition illustrates a first advantage of π_1^τ . Even though it is not a direct consequence of part 1. of Proposition 3.3, the proof is essentially the same.

Proposition 3.19. *For any based spaces X, Y , there is a natural isomorphism $\phi : \pi_1^\tau(X \times Y) \rightarrow \pi_1^\tau(X) \times \pi_1^\tau(Y)$ of topological groups.*

Proof. The projections $X \times Y \rightarrow X$ and $X \times Y \rightarrow Y$ induce the continuous group isomorphism $\phi : \pi_1^\tau(X \times Y) \rightarrow \pi_1^\tau(X) \times \pi_1^\tau(Y)$ given by $\phi([\alpha, \beta]) = ([\alpha], [\beta])$. Let x_0 and y_0 be the basepoints of X and Y respectively and $c_{x_0} : I \rightarrow X$ and $c_{y_0} : I \rightarrow Y$ be the constant loops. The maps $i : X \rightarrow X \times Y, x \mapsto (x, y_0)$ and $j : Y \rightarrow X \times Y, y \mapsto (x_0, y)$ induce the continuous homomorphisms $i_* : \pi_1^\tau(X) \rightarrow \pi_1^\tau(X \times Y)$, $i_*([\alpha]) = [(\alpha, c_{y_0})]$ and $j_* : \pi_1^\tau(Y) \rightarrow \pi_1^\tau(X \times Y)$, $j_*([\beta]) = [(c_{x_0}, \beta)]$. Let μ be group multiplication of the topological group $\pi_1^\tau(X \times Y)$. The composition $\mu \circ (i_* \times j_*)$ is the continuous inverse of ϕ . \square

The author does not know if π_1^τ preserves arbitrary products, however, there is a positive answer when the factor spaces have discrete fundamental groups.

Example 3.20. Let $X_n, n \geq 1$ be a countable family of spaces such that $\pi_1^\tau(X_n)$ is discrete (e.g. X_n a polyhedron or manifold) for each n . Since the quotient map $\pi_n : \Omega(X_n) \rightarrow \pi_1^{qtop}(X_n) = \pi_1^\tau(X_n)$ is open for each $n \geq 1$, both vertical maps in the diagram

$$\begin{array}{ccc} \Omega(X) & \xrightarrow{\cong} & \prod_{n \geq 1} \Omega(X_n) \\ \pi \downarrow & & \downarrow \prod_{n \geq 1} \pi_n \\ \pi_1^{qtop}(X) = \pi_1^\tau(X) & \xrightarrow{\cong} & \prod_{n \geq 1} \pi_1^{qtop}(X_n) = \prod_{n \geq 1} \pi_1^\tau(X_n) \end{array}$$

are quotient. The top (resp. bottom) map is the canonical homeomorphism (resp. group isomorphism). It follows from the universal property of quotient spaces that $\pi_1^\tau(X)$ is a countable product of discrete groups and is therefore a zero-dimensional, metrizable topological group. Moreover, if $\pi_1(X_n) \neq 1$ for infinitely many n , $\pi_1^\tau(X)$ is not discrete. For instance, $\pi_1^\tau(\prod_{n \geq 1} S^1)$ is isomorphic to the countable product $\prod_{n \geq 1} \mathbb{Z}$.

3.4 Separation and the first shape group

Characterizing the spaces X for which $\pi_1^\tau(X)$ is Hausdorff is a non-trivial task. Since every Hausdorff topological group is functionally Hausdorff, the continuity of the identity $\pi_1^{qtop}(X) \rightarrow \pi_1^\tau(X)$ offers an obvious necessary condition:

Proposition 3.21. *If $\pi_1^\tau(X)$ is Hausdorff, then $\pi_1^{qtop}(X)$ is functionally Hausdorff.*

The T_1 axiom in $\pi_1^{qtop}(X)$ is thus necessary for $\pi_1^\tau(X)$ to be Hausdorff but cannot be sufficient since there are spaces Y such that $\pi_1^{qtop}(Y)$ is T_1 but not functionally Hausdorff [4, Example 4.13]. The converse of Proposition 3.21 is true in the special case of Corollary 4.3. It remains an open question whether or not the converse holds in full generality.

In the pursuit of practical necessary conditions for the existence of separation axioms, we recall the notion of a “homotopically path-Hausdorff space” introduced in [17]. This notion is stronger than the notion of “homotopically Hausdorff space” useful for studying generalized universal coverings of locally path connected spaces [18].

Definition 3.22. A space X is *homotopically path-Hausdorff* if given any paths $\alpha, \beta : [0, 1] \rightarrow X$ such that $\alpha(0) = \beta(0)$, $\alpha(1) = \beta(1)$, and α and β are not homotopic rel. endpoints, then there exist a partition $0 = t_0 < t_1 < t_2 < \dots < t_k = 1$ and open sets $U_1, U_2, \dots, U_k \subseteq X$ with $\alpha([t_{i-1}, t_i]) \subset U_i$ such that for any other path $\gamma : [0, 1] \rightarrow X$ satisfying $\gamma(t_i) = \alpha(t_i)$ for $i = 0, \dots, k$ and $\gamma([t_{i-1}, t_i]) \subset U_i$ for $i = 1, \dots, k$, γ is not homotopic to β rel. endpoints.

The following Proposition gives that the homotopically path-Hausdorff property is necessary for the Hausdorff separation axiom in our topologized fundamental group. The homotopically path-Hausdorff property is even more closely related to the T_1 axiom in quasitopological fundamental groups; this connection, studied by Paul Fabel and the author, will be explored further in a future paper.

Proposition 3.23. *If X is path connected and $\pi_1^\tau(X)$ is Hausdorff, then X is homotopically path-Hausdorff.*

Proof. Suppose $\pi_1^\tau(X)$ is Hausdorff. Proposition 3.15 gives that $\pi_1^\tau(X, x)$ is Hausdorff for all $x \in X$. Suppose $\alpha, \beta : [0, 1] \rightarrow X$ are paths such that $\alpha(0) = \beta(0)$, $\alpha(1) = \beta(1)$, but which are not homotopic rel. endpoints. Thus $L = \alpha * \beta^{-1}$ is a non-trivial loop based at $\alpha(0)$.

Since $\pi_1^\tau(X, \alpha(0))$ is Hausdorff, there is an open neighborhood U of $[L]$ which does not contain the identity $[c_{\alpha(0)}]$. Since $\pi : \Omega(X, \alpha(0)) \rightarrow \pi_1^\tau(X, \alpha(0))$ is continuous, $\pi^{-1}(U)$ is an open neighborhood of L . Take a positive even integer $n = 2k$ and open neighborhoods U_1, \dots, U_n such that $\mathcal{U} = \bigcap_{j=1}^n \left(\left[\frac{j-1}{n}, \frac{j}{n} \right], U_j \right)$ is an open neighborhood of L contained in $\pi^{-1}(U)$. Clearly \mathcal{U} contains only non-trivial loops.

Let $t_i = \frac{i}{k}$ for each $i = 0, 1, \dots, k$ and note that $\alpha([t_{i-1}, t_i]) \subset U_i$ for $i = 1, \dots, k$. Suppose $\gamma : [0, 1] \rightarrow X$ is a path satisfying $\gamma(t_i) = \alpha(t_i)$ for each $i = 0, \dots, k$ and $\gamma([t_{i-1}, t_i]) \subset U_i$ for $i = 1, \dots, k$. Now the concatenation $\gamma * \beta^{-1}$ lies in \mathcal{U} and therefore must be non-trivial. Thus γ and β are not homotopical relative to their endpoints. \square

In the search for conditions sufficient to guarantee separation, we turn to shape theory. One is often interested in whether or not the fundamental group of a space X with complicated local structure naturally injects into its first shape homotopy group since such an embedding provides a convenient way to understand the fundamental group. It turns out that this property is strong enough to guarantee that $\pi_1^\tau(X)$ is Hausdorff. We refer to [28] for the preliminaries of shape theory.

The shape group as a topological group 3.24. One way to see that $\pi_1^\tau(X)$ is very often Hausdorff is to use the natural topology on the first shape homotopy group. The homotopy category of polyhedra \mathbf{hPol}_* is the full-subcategory of \mathbf{hTop}_* consisting of spaces with the homotopy type of a polyhedron. It is well-known that \mathbf{hPol}_* is a dense subcategory of \mathbf{hTop}_* [28, §4.3, Theorem 7]. This means that for every based space X , there is an \mathbf{hPol}_* -expansion $X \rightarrow (X_\Lambda, p_{\Lambda\Lambda'}, \Lambda)$ (for instance, the Čech expansion). Specifically, the expansion consists of polyhedra $X_\lambda, \lambda \in \Lambda$ and maps $p_\lambda : X \rightarrow X_\lambda$ such that p_λ is homotopic to $p_{\lambda\lambda'} \circ p_{\lambda'}$ whenever $\lambda' \geq \lambda$ in the directed set Λ . The induced continuous homomorphisms $(p_\lambda)_* : \pi_1^\tau(X) \rightarrow \pi_1^\tau(X_\lambda)$ satisfy $(p_\lambda)_* = (p_{\lambda\lambda'})_* \circ (p_{\lambda'})_*$.

The *first homotopy pro-group* of a based space is the inverse system $pro\text{-}\pi_1(X) = (\pi_1^\tau(X_\lambda), (p_{\lambda\lambda'})_*)$ of discrete groups in $\mathbf{pro}\text{-}\mathbf{TopGrp}$ where the bonding maps are the homomorphisms $(p_{\lambda\lambda'})_* : \pi_1^\tau(X_{\lambda'}) \rightarrow \pi_1^\tau(X_\lambda)$. The *first shape homotopy group* of X is the limit $\tilde{\pi}_1(X) = \varprojlim pro\text{-}\pi_1(X)$ which, as an inverse limit of discrete groups, is a Hausdorff topological group. The continuous homomorphisms $(p_\lambda)_*$ induce a natural, continuous homomorphism $\phi : \pi_1^\tau(X) \rightarrow \tilde{\pi}_1(X)$.

When ϕ is injective, X is said to be π_1 -*injective*. This property is particularly useful since the fundamental group of a π_1 -injective space can be identified as a subgroup of $\tilde{\pi}_1(X)$. Some recent results on π_1 -injectivity include [8, 11, 15, 16].

Proposition 3.25. *If X is π_1 -injective, then $\pi_1^\tau(X)$ is Hausdorff.*

Proof. Any topological group continuously injecting into a Hausdorff group is Hausdorff. \square

The converse of Proposition 3.25 is false in general: See [16, Example 2.4] for a simple counterexample. It remains open whether or not the converse holds for locally path connected spaces.

Example 3.26. For each integer $n \geq 1$, let $C_n = \left\{ (x, y) \in \mathbb{R}^2 \mid \left(x - \frac{1}{n}\right)^2 + y^2 = \frac{1}{n^2} \right\}$ so that $\mathbb{HIE} = \bigcup_{n \geq 1} C_n$ is the usual Hawaiian earring. The shape group $\tilde{\pi}_1(\mathbb{HIE})$ is the inverse limit $\varprojlim_n F_n$ of discrete free groups; here F_n is the free topological group on the discrete space of cardinality n . The canonical homomorphism $\phi : \pi_1(\mathbb{HIE}) \rightarrow \varprojlim_n F_n$ is injective [31] and therefore $\pi_1^\tau(\mathbb{HIE})$ is Hausdorff. Since $\pi_1^\tau(\mathbb{HIE})$ is a topological group, it is necessarily Tychonoff. Currently, it is not even known if the quasitopological fundamental group $\pi_1^{qtop}(\mathbb{HIE})$ is regular.

Though the homomorphism ϕ is injective, it is known that $\phi : \pi_1^{qtop}(\mathbb{HIE}) \rightarrow \varprojlim_n F_n$ fails to be a topological embedding [12]. Since $\pi_1^{qtop}(\mathbb{HIE})$ is not a topological group [13], the topology of $\pi_1^\tau(\mathbb{HIE})$ is strictly coarser than the quotient topology of $\pi_1^{qtop}(\mathbb{HIE})$. From this fact alone, it is plausible that $\phi : \pi_1^\tau(\mathbb{HIE}) \rightarrow \varprojlim_n F_n$ is an embedding.

The semicovering theory [6] of $\mathbb{H}\mathbb{E}$, however, can be used to show that $\phi : \pi_1^\tau(\mathbb{H}\mathbb{E}) \rightarrow \varprojlim_n F_n$ also fails to be an embedding. Indeed, if this map is an embedding, the normal, open subgroups $K_n = \ker(\pi_1^\tau(\mathbb{H}\mathbb{E}) \rightarrow F_n)$ form a neighborhood base at the identity of $\pi_1^\tau(\mathbb{H}\mathbb{E})$. A simple consequence of such a basis is: An subgroup H is open in $\pi_1^\tau(\mathbb{H}\mathbb{E})$ if and only if there is a covering map $p : Y \rightarrow X$ such that $p_*(\pi_1(Y)) = H$. But this cannot be the case since open subgroups of $\pi_1^\tau(\mathbb{H}\mathbb{E})$ are classified by semicoverings of $\mathbb{H}\mathbb{E}$ and there are examples of semicovering maps $p : Y \rightarrow \mathbb{H}\mathbb{E}$ which are not covering maps [6, Example 3.8].

This means we have three distinct natural topologies on $\pi_1(\mathbb{H}\mathbb{E})$ from finest to coarsest: the quotient topology, the topology of $\pi_1^\tau(\mathbb{H}\mathbb{E})$, and the initial topology with respect to $\phi : \pi_1(\mathbb{H}\mathbb{E}) \rightarrow \varprojlim_n F_n$.

Example 3.27. The map $\phi : \pi_1^\tau(X) \rightarrow \tilde{\pi}_1(X)$ also fails to be an embedding in simple non-locally path connected cases. Consider the generalized wedge $\Sigma(X_+)$ in Example 4.5 below where X is the one-point compactification of the natural numbers. The free topological group $\pi_1^\tau(\Sigma(X_+)) \cong F_M(X)$ is not first countable [1, 7.1.20] but since $\Sigma(X_+)$ is a planar continuum, the shape group $\tilde{\pi}_1(\Sigma(X_+))$ is metrizable and $\phi : \pi_1(\Sigma(X_+)) \rightarrow \tilde{\pi}_1(\Sigma(X_+))$ is injective [15]. Therefore the topology of $\pi_1^\tau(\Sigma(X_+))$ is strictly finer than the initial topology with respect to ϕ .

Despite the failure of embedding in the previous two examples, there are simple examples of non-semilocally simply connected spaces whose fundamental group embeds in its first shape group. For instance, if $X = \prod_{n \geq 1} S^1$ is the countable product of circles, as in Example 3.20, then $\phi : \pi_1^\tau(X) \rightarrow \tilde{\pi}_1(X)$ is an isomorphism of non-discrete metrizable groups. These examples motivate the following general question.

Question 3.28. For which spaces X is $\phi : \pi_1^\tau(X) \rightarrow \tilde{\pi}_1(X)$ a topological embedding?

4 Main Results: Realizing topological groups as fundamental groups

The fact that one can conveniently realize a free group or a free product of groups as the fundamental group of a space is quite useful, for instance, in proving the Nielsen–Schreier theorem and Kurosh subgroup theorem. The main results of the current paper, laid out in the following three sections, are meant to be such “realization results” for topological groups.

4.1 Generalized wedges of circles and free topological groups

The ability of π_1^τ to distinguish spaces with isomorphic fundamental groups can be observed by making a computation analogous to the elementary fact that the fundamental group of a wedge of circles indexed by a set X is free on X . The *generalized wedge of circles on (an unbased space) X* is constructed as the reduced suspension $\Sigma(X_+)$ of the space $X_+ = X \sqcup \{*\}$ with isolated basepoint. Equivalently, it is the quotient space $X \times I / X \times \{0, 1\}$. The image of $(x, t) \in X \times I$ in the quotient is written as $x \wedge t$. Similarly, $A \wedge B$ denotes the image of $A \times B \subset X \times I$. The canonical basepoint is $x_0 = X \wedge \{0, 1\}$. Since $\Sigma(X_+) \cong \bigvee_X S^1$ whenever X is a discrete space, this construction clearly generalizes the construction of a wedge of circles.

The main result in this section is that the fundamental group of a generalized wedge of circles $\Sigma(X_+)$ is the free topological group on the path component space of X . The proof calls upon the technical computation of $\pi_1^{qtop}(\Sigma(X_+))$ in [4].

Theorem 4.1. *There is a natural isomorphism $h_X : F_M(\pi_0^{qtop}(X)) \rightarrow \pi_1^\tau(\Sigma(X_+))$ of topological groups.*

Proof. The unit $u : X \rightarrow \Omega(\Sigma(X_+))$ of the adjunction $\mathbf{Top}_*(\Sigma(X_+), Y) \cong \mathbf{Top}(X, \Omega Y)$ induces a continuous injection $u_* : \pi_0^{qtop}(X) \rightarrow \pi_1^{qtop}(\Sigma(X_+))$ on path component spaces. One of the main results in [4] is that for arbitrary X , u_* induces a natural group isomorphism $h_X : F(\pi_0(X)) \rightarrow \pi_1(\Sigma(X_+))$ so that $h_X^{-1} \circ u_*$ is the canonical injection of generators and $h_X^{-1} : \pi_1^{qtop}(\Sigma(X_+)) \rightarrow F_M(\pi_0^{qtop}(X))$ is continuous. Since $F_M(\pi_0^{qtop}(X))$ is a topological group, $h_X^{-1} : \pi_1^\tau(\Sigma(X_+)) \rightarrow F_M(\pi_0^{qtop}(X))$ is continuous. Additionally, the continuous injection $u_* : \pi_0^{qtop}(X) \rightarrow \pi_1^{qtop}(\Sigma(X_+)) \rightarrow \pi_1^\tau(\Sigma(X_+))$ induces (by the universal property of free topological groups) the continuous inverse $h_X : F_M(\pi_0^{qtop}(X)) \rightarrow \pi_1^\tau(\Sigma(X_+))$. \square

Prior to studying some special cases of generalized wedges, we apply the theory of free topological groups in two Corollaries.

Corollary 4.2. *A quotient map (resp. homotopy equivalence) $f : X \rightarrow Y$ induces a quotient map (resp. an isomorphism) $f_* : \pi_1^\tau(\Sigma(X_+)) \rightarrow \pi_1^\tau(\Sigma(Y_+))$ of topological groups.*

Proof. This follows directly from the fact that both the functors F_M and π_0^{top} preserve quotients and that π_0^{top} takes homotopy equivalences to homeomorphisms. \square

Corollary 4.3. *For any unbased space X , the following are equivalent:*

1. $\pi_1^\tau(\Sigma(X_+))$ is Hausdorff.
2. $\pi_1^{qtop}(\Sigma(X_+))$ is functionally Hausdorff.
3. $\pi_0^{qtop}(X)$ is functionally Hausdorff.

Proof. 1. \Rightarrow 2. is a case of Proposition 3.21. 2. \Rightarrow 3. follows from the fact that $u_* : \pi_0^{qtop}(X) \rightarrow \pi_1^{qtop}(\Sigma(X_+))$ is a continuous injection. 3. \Rightarrow 1. follows from Theorem 4.1 and 1. of Lemma 3.1. \square

Note that when $\dim(X) = 0$, or more generally when X is totally path disconnected, the isomorphism of Theorem 4.1 simplifies to $\pi_1^\tau(\Sigma(X_+)) \cong F_M(X)$. In this case, $\Sigma(X_+)$ provides a particularly nice geometric interpretation of $F_M(X)$.

Remark 4.4. In the case $X = \Omega(Y)$, the counit $\Sigma(\Omega(Y)_+) \rightarrow Y$ induces the multiplication map $m_{\pi_1^{qtop}(Y)} : F_M(\pi_1^{qtop}(Y)) \cong \pi_1^\tau(\Sigma(\Omega(Y)_+)) \rightarrow \pi_1^\tau(Y)$ used to define the topology of $\pi_1^\tau(Y)$.

Example 4.5. Let ω be the discrete space of natural numbers. We find an interesting case when $X = \{1, 2, \dots, \infty\}$ is the one-point compactification of the natural numbers. We write $X = \omega + 1$ when we wish to view X as the first compact, infinite ordinal. The generalized wedge $\Sigma(X_+)$ is homeomorphic to the planar continuum

$$S^1 \cup \bigcup_{n \geq 1} \left\{ (x, y) \in \mathbb{R}^2 \mid \left(x - \frac{1}{n}\right)^2 + y^2 = \left(1 + \frac{1}{n}\right)^2 \right\}$$

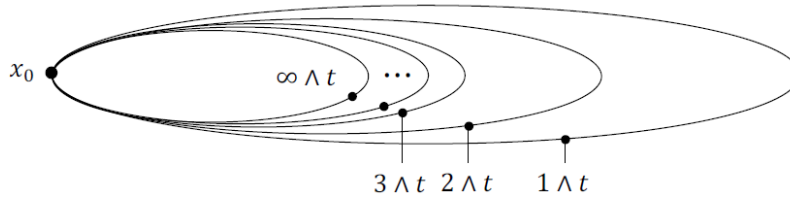


Figure 1: The generalized wedge on the one-point compactification of the natural numbers.

Let dX be the underlying set of X with the discrete topology. The identity $dX \rightarrow X$ induces a weak equivalence $\bigvee_{dX} S^1 \cong \Sigma((dX)_+) \rightarrow \Sigma(X_+)$, but $\pi_1^\tau(\bigvee_{dX} S^1)$ is discrete by Corollary 3.17 and $\pi_1^\tau(\Sigma(X_+)) \cong F_M(\omega + 1)$ is not discrete. Therefore $\Sigma(X_+)$ is not homotopy equivalent to a countable wedge of circles.

One can generalize this ability to distinguish homotopy types of one-dimensional spaces with isomorphic fundamental groups by noting that if two zero-dimensional spaces X and Y have the same cardinality but are such that $F_M(X) \not\cong F_M(Y)$, then $\pi_1(\Sigma(X_+)) \cong F(X) \cong F(Y) \cong \pi_1(\Sigma(Y_+))$ but $\pi_1^\tau(\Sigma(X_+)) \not\cong \pi_1^\tau(\Sigma(Y_+))$.

Let X and Y be countably infinite compact Hausdorff spaces. Any such space is homeomorphic to a countable successor ordinal (with the order topology) by the Mazurkiewicz-Sierpinski Theorem [30] and embeds into \mathbb{Q} . Thus $\Sigma(X_+)$ and $\Sigma(Y_+)$ are one-dimensional (but non-Peano) planar continua with $\pi_1(\Sigma(X_+)) \cong F(X) \cong F(Y) \cong \pi_1(\Sigma(Y_+))$. Baars [2] gives the following characterization using a result of Graev [23]: $F_M(X)$ and $F_M(Y)$ are isomorphic topological groups if and only if there are countable ordinals α, β such that $X \cong \alpha + 1$, $Y \cong \beta + 1$, and $\max(\alpha, \beta) < (\min(\alpha, \beta))^\omega$. This result is put to use in the next example.

Example 4.6. If $X = \omega + 1$ and $Y = \omega^\omega + 1$, then $\Sigma(X_+)$ and $\Sigma(Y_+)$ are one-dimensional (but non-Peano) planar continua. Though these spaces have isomorphic fundamental groups (free on countable generators), they cannot be homotopy equivalent. Indeed, if α and β are ordinals homeomorphic to ω and ω^ω respectively, then we necessarily have $\alpha = \omega$ and $\beta = \omega^\omega$ [26]. But $\max(\alpha, \beta) = \omega^\omega = (\min(\alpha, \beta))^\omega$ and therefore $\pi_1^\tau(\Sigma(X_+))$ and $\pi_1^\tau(\Sigma(Y_+))$ are non-isomorphic, non-discrete, Hausdorff topological groups.

4.2 Every topological group is a fundamental group

The fact that every group is realized as a fundamental group is easily arrived at by the process of attaching 2-cells to wedges of circles. Similarly, we attach 2-cells to generalized wedges of circles (Section 4.1) to realize every topological group as the fundamental group of some space. First, we note that attaching n -cells to a space changes the topology of the fundamental group in a convenient way. The following Lemma first appeared in [3]; an alternative proof appears in [4].

Lemma 4.7. Suppose Z is a based space, $n \geq 2$ an integer, and $f : S^{n-1} \rightarrow Z$ is a based map. Let $Z' = Z \sqcup_f e^n$ be the space obtained by attaching an n -cell to Z via the attaching map f . The inclusion $j : Z \hookrightarrow Z'$ induces a homomorphism $j_* : \pi_1^{qtop}(Z) \rightarrow \pi_1^{qtop}(Z')$ which is also a topological quotient map.

The compactness of S^1 allows us to easily extend Lemma 4.7 to an arbitrary number of cells.

Lemma 4.8. Suppose Z is a based space, $n \geq 2$ an integer, and $f_\alpha : S^{n-1} \rightarrow Z$, $\alpha \in A$ is a family of based maps. Let $Z' = Z \sqcup_{f_\alpha} e_\alpha^n$ be the space obtained by attaching n -cells to Z via the attaching maps f_α . The inclusion $j : Z \hookrightarrow Z'$ induces a homomorphism $j_* : \pi_1^{qtop}(Z) \rightarrow \pi_1^{qtop}(Z')$ which is also a topological quotient map.

Proof. We re-label $Z = Z_1$ and $Z' = Z_4$ and take the approach of factoring the inclusion $j : Z_1 \hookrightarrow Z_4$ twice as $Z_1 \subseteq Z_2 \subseteq Z_3 \subseteq Z_4$. In general, $\pi_k : \Omega(Z_k) \rightarrow \pi_1^{qtop}(Z_k)$ denotes the quotient map identifying homotopy classes of maps.

Consider the commutative diagram

$$\begin{array}{ccc} \Omega(Z_1) & \xrightarrow{j_\#} & \Omega(Z_4) \\ \pi_1 \downarrow & & \downarrow \pi_4 \\ \pi_1^{qtop}(Z_1) & \xrightarrow{j_*} & \pi_1^{qtop}(Z_4) \end{array}$$

and suppose $U \subseteq \pi_1^{qtop}(Z_4)$ such that $j_*^{-1}(U)$ is open in $\pi_1^{qtop}(Z_1)$. It suffices to show that $\pi_4^{-1}(U)$ is open in $\Omega(Z_4)$. Let $\beta \in \pi_4^{-1}(U)$. Since the image $\beta(S^1)$ is compact, it intersects only finitely many of the attached cells. Suppose $\alpha_1, \dots, \alpha_N$ are the indices in A such that $\beta(S^1) \cap e_{\alpha_i}^n \neq \emptyset$. Let $Z_2 = Z_1 \sqcup_{\alpha_i} e_{\alpha_i}^n \subseteq Z_4$ be the subspace of Z_4 which is Z_1 with the cells $e_{\alpha_1}^n, \dots, e_{\alpha_N}^n$ attached. Additionally, for each $\alpha \in A - \{\alpha_1, \dots, \alpha_N\}$, take a point $z_\alpha \in \text{int}(e_\alpha^n)$ and let $Z_3 = Z_4 - \{z_\alpha | \alpha \in A - \{\alpha_1, \dots, \alpha_N\}\}$ be the open subspace of Z_4 with the chosen interior points removed. We know from Lemma 4.7 that the inclusion $j_1 : Z_1 \hookrightarrow Z_2$ induces a quotient map $(j_1)_* : \pi_1^{qtop}(Z_1) \rightarrow \pi_1^{qtop}(Z_2)$ since Z_2 is obtained from Z_1 by attaching only finitely many n -cells. The inclusion $j_2 : Z_2 \hookrightarrow Z_3$ is a homotopy equivalence and therefore induces an isomorphism $(j_2)_* : \pi_1^{qtop}(Z_2) \rightarrow \pi_1^{qtop}(Z_3)$ of quasitopological groups. Lastly, since S^1 is compact and Z_3 is open in Z_4 , the inclusion $j_3 : Z_3 \hookrightarrow Z_4$

induces an open embedding $(j_3)_\# : \Omega(Z_3) \hookrightarrow \Omega(Z_4)$ on loop spaces. Overall, we have $j_3 \circ j_2 \circ j_1 = j$ and that $(j_2 \circ j_1)_* = (j_2)_* \circ (j_1)_* : \pi_1^{qtop}(Z_1) \rightarrow \pi_1^{qtop}(Z_3)$ is a quotient map. The equality

$$j_*^{-1}(U) = (j_2 \circ j_1)_*^{-1}((j_3)_*^{-1}(U))$$

then implies that $(j_3)_*^{-1}(U)$ is open in $\pi_1^{qtop}(Z_3)$. Therefore, $V = \pi_3^{-1}((j_3)_*^{-1}(U)) = (j_3)_\#^{-1}(\pi_4^{-1}(U))$ is an open neighborhood of β in $\Omega(Z_3)$. Since $(j_3)_\# : \Omega(Z_3) \hookrightarrow \Omega(Z_4)$ is an open embedding, $(j_3)_\#(V)$ is an open neighborhood of β in $\Omega(Z_4)$. If $\gamma \in (j_3)_\#(V)$, then we have a loop $\gamma' \in V$ such that $j_3 \circ \gamma' = \gamma$. But this means $[\gamma'] \in (j_3)_*^{-1}(U)$, so that $[\gamma] = [j_3 \circ \gamma'] \in U$ and consequently $\gamma \in \pi_4^{-1}(U)$. This gives the inclusion $(j_3)_\#(V) \subseteq \pi_4^{-1}(U)$ and that $\pi_4^{-1}(U)$ is open in $\Omega(Z_4)$. \square

Since τ preserves quotient maps (Proposition 3.3), we have:

Corollary 4.9. *Suppose Z is a based space, $n \geq 2$ an integer, and $f_\alpha : S^{n-1} \rightarrow Z$, $\alpha \in A$ is a family of based maps. Let $Z' = Z \sqcup_{f_\alpha} e_\alpha^n$ be the space obtained by attaching n -cells to Z via the attaching maps f_α . The inclusion $j : Z \hookrightarrow Z'$ induces a quotient map $j_* : \pi_1^\tau(Z) \rightarrow \pi_1^\tau(Z')$ of topological groups.*

We use this Corollary and the realization of free topological groups as fundamental groups (Theorem 4.1) to construct, for any topological group G , a space Z whose fundamental group is isomorphic to G .

Theorem 4.10. *Every topological group G is isomorphic to the fundamental group $\pi_1^\tau(Z)$ of a space Z obtained by attaching 2-cells to a generalized wedge of circles $\Sigma(X_+)$. Moreover, one may continue to attach cells of dimension > 2 to obtain a space Z' such that $\pi_1^\tau(Z') \cong G$ and $\pi_n^\tau(Z') = 0$ for $n > 1$.*

Proof. Suppose G is a topological group. According to the main result of [24], there is a (paracompact Hausdorff) space X such that $\pi_0^{qtop}(X)$ is homeomorphic to the underlying space of G . The construction of X is functorial and the homeomorphism $\pi_0^{qtop}(X) \cong G$ is natural, however, X does not seem to inherit any natural algebraic structure. In the case that G is totally path disconnected, take $X = G$. This gives natural isomorphisms

$$h_G : \pi_1^\tau(\Sigma(X_+)) \cong F_M(\pi_0^{qtop}(X)) \cong F_M(G).$$

The identity $G \rightarrow G$ induces the quotient map $m_G : F_M(G) \rightarrow G$ so that $m_G \circ h_G : \pi_1^\tau(\Sigma(X_+)) \cong F_M(G) \rightarrow G$ is a quotient map of topological groups. For each $g \in G \cong \pi_0^{qtop}(X)$ fix a point $x_g \in X$ such that the path component of x_g corresponds to g . Recall that $u_{x_g} : I \rightarrow \Sigma(X_+)$ is the loop $u_{x_g}(t) = x_g \wedge t$ and the set of homotopy classes $\{[u_{x_g}] | g \in G\}$ freely generate $\pi_1(\Sigma(X_+)) \cong F(G)$. For each $\alpha \in \ker(m_G \circ h_G)$ choose a representative loop $f_\alpha = u_{x_{g_1}} * u_{x_{g_2}} * \dots * u_{x_{g_n}} : I \rightarrow \Sigma(X_+)$ and attach a 2-cell to $\Sigma(X_+)$ via f_α . If $Z = \Sigma(X_+) \sqcup_{f_\alpha} e_\alpha^2$ is the resulting space, Corollary 4.9 gives that the inclusion $j : \Sigma(X_+) \hookrightarrow Z$ induces a quotient map $j_* : \pi_1^\tau(\Sigma(X_+)) \rightarrow \pi_1^\tau(Z)$. Since $\ker(m_G \circ h_G) = \ker j_*$ and both $m_G \circ h_G$ and j_* are quotient, $\pi_1^\tau(Z) \cong G$ as topological groups.

The second statement of the Theorem follows by the usual process of inductively killing the n -th homotopy group by attaching cells of dimension $n + 1$. The fact that the inclusion at each step induce group isomorphisms on the fundamental group (which are topological quotients by Lemma 4.8 and therefore homeomorphisms) means the direct limit space Z' will satisfy $\pi_1^\tau(Z') \cong G$ and $\pi_n^\tau(Z') = 0$ for all $n > 1$. \square

In the construction of Z' in the previous theorem one will notice that Z' is a CW-complex (and therefore a proper $K(G, 1)$) if and only if $X = G$ is a discrete group. This theorem also permits the odd phenomenon of taking non-trivial fundamental groups of fundamental groups. For instance, there is a space X such that $\pi_1^\tau(X) \cong S^1$ and thus $\pi_1^\tau(\pi_1^\tau(X)) \cong \mathbb{Z}$.

4.3 A topological van Kampen theorem

In this section, we prove a computational result analogous to the classical Seifert-van Kampen Theorem for fundamental groups. The many variations of the van Kampen theorem are also likely to have topological analogues. Consider the following general statement.

Statement 4.11. Let (X, x_0) be a based space and $\{U_1, U_2, U_1 \cap U_2\}$ be an open cover of X consisting of path connected neighborhoods each containing x_0 . The diagram

$$\begin{array}{ccc} \pi_1^\tau(U_1 \cap U_2) & \longrightarrow & \pi_1^\tau(U_1) \\ \downarrow & & \downarrow \\ \pi_1^\tau(U_2) & \longrightarrow & \pi_1^\tau(X) \end{array}$$

induced by inclusions is a pushout in the category of topological groups.

Unfortunately, this statement does not hold in full generality.

Example 4.12. Let $X = \{1, 2, \dots\} \cup \{\infty\}$ be the one-point compactification of the discrete space of natural numbers and $\Sigma(X_+)$ be the generalized wedge of Example 4.5. A basic open neighborhood of a point $x \wedge t \in \Sigma(X_+) - \{x_0\}$ is $U \wedge (a, b) = \{u \wedge s \mid u \in U, s \in S\}$ where U is an open neighborhood of x in X and $t \in (a, b) \subseteq (0, 1)$. The contractible subspaces $X \wedge ([0, \epsilon) \cup (1 - \epsilon, 1])$, $\epsilon < \frac{1}{2}$ form a neighborhood base at the canonical basepoint x_0 . We construct a space Y by attaching 1-cells to $\Sigma(X_+)$. For each $x \in X$, let $f_x : S^0 \rightarrow \Sigma(X_+)$ be the map given by $f_x(-1) = x_0$ and $f_x(1) = x \wedge \frac{1}{2}$. Let $Y = \Sigma(X_+) \sqcup_{f_x} e_x^1$ be the space obtained by attaching a copy of the interval $e_x^1 = [-1, 1]$ for each x via the attaching map f_x .

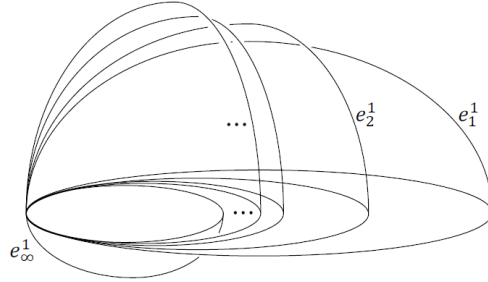


Figure 2: $Y = \Sigma(X_+) \sqcup_{f_x} e_x^1$

Note that any open neighborhood of the loop $\alpha : I \rightarrow \Sigma(X_+) \subset Y$, $\alpha(t) = \infty \wedge t$ contains loops which are not homotopic to α . Thus $\pi_1^\tau(Y)$ is not discrete by Proposition 3.17. Define an open cover of Y by letting

$$U_1 = \left(X \wedge \left(\left[0, \frac{1}{6}\right) \cup \left(\frac{2}{6}, 1\right] \right) \right) \cup \bigcup_{x \in X} e_x^1 \text{ and } U_2 = \left(X \wedge \left(\left(\frac{5}{6}, 1\right] \cup \left[0, \frac{4}{6}\right) \right) \right) \cup \bigcup_{x \in X} e_x^1$$

Note that $U_1 \cong U_2$.

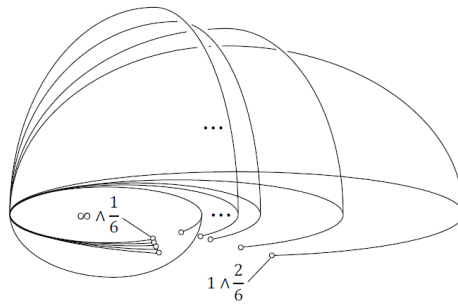


Figure 3: The open set $U_1 \subset Y$

Collapsing the set $\Sigma(X_+) \cap U_1$ to a point gives map $U_1 \rightarrow \bigvee_X S^1$ to a countable wedge of circles which induces an isomorphism $\pi_1^\tau(U_1) \rightarrow \pi_1^\tau(\bigvee_X S^1)$ of discrete topological groups. Consequently, $\pi_1^\tau(U_1) \cong \pi_1^\tau(U_2)$ is the discrete free group on countably many generators. Note that

$$U_1 \cap U_2 = \left(X \wedge \left[0, \frac{1}{6}\right) \cup \left(\frac{2}{6}, \frac{4}{6}\right) \cup \left(\frac{5}{6}, 1\right] \right) \cup \bigcup_{x \in X} e_x^1$$

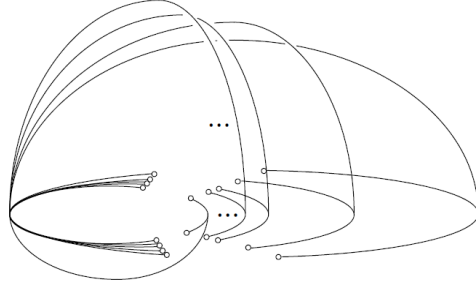


Figure 4: $U_1 \cap U_2$

Clearly $\pi_1(U_1 \cap U_2) = 1$. If the square

$$\begin{array}{ccc} \pi_1^\tau(U_1 \cap U_2) & \longrightarrow & \pi_1^\tau(U_1) \\ \downarrow & & \downarrow \\ \pi_1^\tau(U_2) & \longrightarrow & \pi_1^\tau(Y) \end{array}$$

is a pushout in the category of topological groups, then $\pi_1^\tau(Y)$ is the free topological product of two discrete groups and must also be discrete. This contradiction indicates that Statement 4.11 cannot be true in full generality.

The complication arising in the previous example motivates the following definition.

Definition 4.13. A path $p : I \rightarrow X$ is *well-ended* if for every open neighborhood \mathcal{U} of p in $P(X)$ there are open neighborhoods V_0, V_1 of $p(0), p(1)$ in X respectively such that for every $a \in V_0, b \in V_1$ there is a path $q \in \mathcal{U}$ with $q(0) = a$ and $q(1) = b$. A space X is *wep-connected* if for every $x, y \in X$, there is a well-ended path from x to y .

Remark 4.14. Since a basis for the topology of $P(X)$ is given by neighborhoods of the form $\bigcap_{j=1}^n \langle K_n^j, U_j \rangle$ where U_j is open in X , we have: A path $\alpha \in P(X)$ is well-ended if and only if for each neighborhood $\bigcap_{j=1}^n \langle K_n^j, U_j \rangle$ of α , there are open neighborhoods $V_0 \subseteq U_1, V_1 \subseteq U_n$ of $\alpha(0), \alpha(1)$ in X respectively such that for every $a \in V_0, b \in V_1$ there is a path $\beta \in \bigcap_{j=1}^n \langle K_n^j, U_j \rangle$ with $\beta(0) = a$ and $\beta(1) = b$. It is instructive to observe this in the following figure.

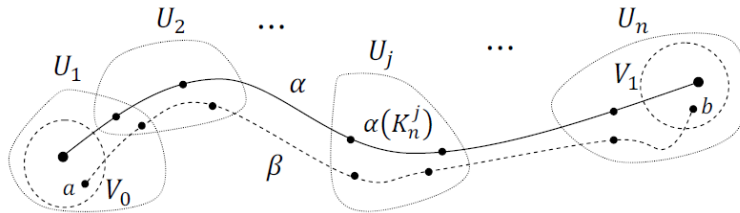


Figure 5: A well-ended path α .

Well ended-paths play an important role in the theory of semicoverings [6]. We avoid an involved discussion of wep-connected spaces by restricting to what is necessary for Theorem 4.23. We first note that wep-connectedness is a natural generalization of local path connectedness.

Proposition 4.15. *If $p : I \rightarrow X$ is a path and X is locally path connected at $p(0)$ and $p(1)$, then p is well-ended. Consequently, all path connected, locally path connected spaces are wep-connected.*

Proof. Suppose $\mathcal{U} = \bigcap_{j=1}^n \langle K_n^j, U_j \rangle$ is a basic open neighborhood of p in $P(X)$. Find a path connected neighborhood V_0, V_1 of $p(0), p(1)$ respectively such that $V_0 \subseteq U_1$ and $V_1 \subseteq U_n$. For points $a \in V_0, b \in V_1$, take paths $\alpha : I \rightarrow V_0$ from a to $p(0)$ and $\beta : I \rightarrow V_1$ from $p(1)$ to b . Now define a path $q \in \mathcal{U}$ by setting

$$q_{K_{2n}^1} = \alpha, q_{K_{2n}^2} = p_{K_n^1}, q_{[\frac{1}{n}, \frac{n-1}{n}]} = p_{[\frac{1}{n}, \frac{n-1}{n}]}, q_{K_{2n}^{2n-1}} = p_{K_n^{n-1}}, \text{ and } q_{K_{2n}^{2n}} = \beta$$

Clearly q is a path in \mathcal{U} from a to b . □

A straightforward partial converse to Proposition 4.15 is that a space X is locally path connected if and only if every constant path $I \rightarrow X$ is well-ended. Many non-locally path connected spaces are wep-connected.

Example 4.16. It is an easy exercise to verify that for every space X , the generalized wedge $\Sigma(X_+)$ is wep-connected; the canonical choice of well-ended paths $I \rightarrow \Sigma(X_+)$ are those which have image in a circle $x \wedge I \cong S^1$ for some $x \in X$. For instance, the generalized wedge in Example 4.5 is wep-connected but is not locally path connected. More generally, spaces, such as Z and Z' in Theorem 4.10, obtained by attaching cells to a generalized wedge are wep-connected but are not typically locally path connected.

It is worthwhile to observe two examples of space which are not wep-connected.

Example 4.17. Zeeman's example [25, 6.6.14] illustrated below is not wep-connected since there are no well-ended paths starting or ending at the two points where the space fails to be locally path connected.

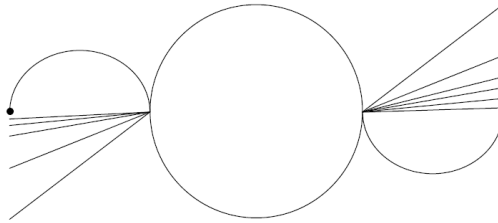


Figure 6: Zeeman's example

Example 4.18. In Example 4.12, the space Y is wep-connected, however, the intersection $U_1 \cap U_2$ is not. It turns out that this is precisely why Statement 4.11 fails to hold for the given choice of U_1 and U_2 . Additionally, this situation illustrates that a path connected, open subspace of a wep-connected space need not be wep-connected.

Remark 4.19. In Definition 4.13, it is necessary to specify the codomain X since it may occur that a path $p : I \rightarrow A$ in a subspace $A \subseteq X$ is well-ended whereas $p : I \rightarrow A \hookrightarrow X$ is not. This complication does not arise when A is an open subset of X since, whenever A is open, $P(A)$ is an open subspace of $P(X)$.

Proposition 4.20. *If A is open in X and $p : I \rightarrow A$ is a path, then $p : I \rightarrow A$ is well-ended if and only if $p : I \rightarrow A \hookrightarrow X$ is well-ended.*

The following lemma is useful since the proof of the van Kampen theorem requires a fixed basepoint.

Lemma 4.21. Fix any $x_0 \in X$. Then X is wep-connected if and only if for each $x \in X$ there is a path $p : I \rightarrow X$ from x_0 to x such that for every open neighborhood \mathcal{U} of p in $P(X, x_0)$, there is an open neighborhood V of x such that for each $v \in V$, there is a path $q \in \mathcal{U}$ from x_0 to v .

Proof. Clearly, if X is wep-connected, the second statement holds. For the converse, pick $a, b \in X$. By assumption there are paths $\alpha, \beta \in P(X, x_0)$ ending at a, b respectively each satisfying the conditions in the statement of the lemma. We claim $\alpha * \beta^{-1}$ is well-ended. Let $\mathcal{U} = \bigcap_{j=1}^n \langle K_n^j, U_j \rangle$ be a basic open neighborhood of $\alpha * \beta^{-1}$ in $P(X)$. Since $\mathcal{A} = \mathcal{U}_{[0, \frac{1}{2}]}$ and $\mathcal{B} = \left(\mathcal{U}_{[\frac{1}{2}, 1]}\right)^{-1} \cap P(X, x_0)$ are open neighborhoods of α and β respectively, there are open neighborhoods A of a and B of b such that for any $a' \in A$ (resp. $b' \in B$) there is a path $\alpha' \in \mathcal{A}$ (resp. $\beta' \in \mathcal{B}$) from x_0 to a' (resp. x_0 to b'). Now $\alpha' * (\beta')^{-1} \in \mathcal{U}$ is the desired path from a' to b' . \square

It is thus convenient to give the following definition.

Definition 4.22. A path $p : I \rightarrow X$ is *well-targeted* if for every open neighborhood \mathcal{U} of p in $P(X, p(0))$, there is an open neighborhood V of $p(1)$ such that for each $v \in V$, there is a path $q \in \mathcal{U}$ from $p(0)$ to v .

Statement 4.11 is now proven in the case that the intersection $U_1 \cap U_2$ is wep-connected.

The van Kampen Theorem 4.23. Let (X, x_0) be a based space and $\{U_1, U_2, U_1 \cap U_2\}$ an open cover of X consisting of path connected neighborhoods each containing x_0 . Let $k_i : U_1 \cap U_2 \hookrightarrow U_i$ and $l_i : U_i \hookrightarrow X$ be the inclusions. If $U_1 \cap U_2$ is wep-connected, the induced diagram of continuous homomorphisms

$$\begin{array}{ccc} \pi_1^\tau(U_1 \cap U_2) & \xrightarrow{(k_1)_*} & \pi_1^\tau(U_1) \\ (k_2)_* \downarrow & & \downarrow (l_1)_* \\ \pi_1^\tau(U_2) & \xrightarrow{(l_2)_*} & \pi_1^\tau(X) \end{array}$$

is a pushout square in the category of topological groups. In other words, there is a canonical isomorphism

$$\pi_1^\tau(X) \cong \pi_1^\tau(U_1) *_{\pi_1^\tau(U_1 \cap U_2)} \pi_1^\tau(U_2)$$

of topological groups.

Proof. We show: If G is a topological group and $f_i : \pi_1^\tau(U_i) \rightarrow G$ are continuous homomorphisms such that $f_1 \circ (k_1)_* = f_2 \circ (k_2)_*$, there is a unique, continuous homomorphism $\Phi : \pi_1^\tau(X) \rightarrow G$ such that $\Phi \circ (l_i)_* = f_i$. The classical van Kampen theorem guarantees the existence and uniqueness of the homomorphism Φ and so it suffices to show Φ is continuous. To do this, we show the composition $\phi = \Phi \circ \pi : \Omega(X) \rightarrow \pi_1^\tau(X) \rightarrow G$ is continuous. If this can be done, Proposition 3.12 guarantees the continuity of $\Phi : \pi_1^\tau(X) \rightarrow G$.

We first recall a convenient description of Φ : Given any loop $\alpha \in \Omega(X)$, find a subdivision $0 = t_0 < t_1 < \dots < t_n = 1$ such that $\alpha_j = \alpha|_{[t_{j-1}, t_j]}$ has image in U_{i_j} for $i_j \in \{1, 2\}$. For $j = 1, \dots, n-1$ find a path $p_j : I \rightarrow U_{i_j}$ from x_0 to $a_j = \alpha(t_j)$. If $a_j \in U_1 \cap U_2$, we demand that p_j has image in $U_1 \cap U_2$. Let $p_0 = p_n = c_{x_0}$ be the constant path and $L_j = p_{j-1} * \alpha_j * p_j^{-1}$ for $j = 1, \dots, n$. Note that L_j has image in U_{i_j} and if α_j has image in $U_1 \cap U_2$, then so does L_j . Additionally, $[\alpha]$ is the product $[L_1] \cdots [L_n]$ in $\pi_1(X)$. Now $\Phi([\alpha])$ is defined to be the product

$$f_{i_1}([L_1])f_{i_2}([L_2]) \cdots f_{i_n}([L_n]).$$

That Φ is a well-defined homomorphism follows from arguments found in most first course algebraic topology textbooks [29]. It is particularly useful to the current proof that the given description of $\Phi([\alpha])$ does not depend upon the choice of subdivision $0 = t_0 < t_1 < \dots < t_n = 1$ or paths p_1, \dots, p_{n-1} .

To see that ϕ is continuous, suppose W is open in G and $\alpha \in \phi^{-1}(W)$. Write $\phi(\alpha) = \Phi([\alpha]) = f_{i_1}([L_1])f_{i_2}([L_2]) \cdots f_{i_n}([L_n])$ as above. In particular, choose the subdivision $0 = t_0 < t_1 < \dots < t_n = 1$ so

that $i_j \neq i_{j+1}$. This gives $a_j \in U_1 \cap U_2$ for each j and thus each p_j has image in $U_1 \cap U_2$. Since p_j has image in wep-connected neighborhood $U_1 \cap U_2$, we may assume the paths p_1, \dots, p_{n-1} are well-targeted.

We construct an open neighborhood \mathcal{U} of α contained in $\phi^{-1}(W)$ by combining neighborhoods of its restrictions. Since G is a topological group and $f_{i_1}([L_1])f_{i_2}([L_2])\dots f_{i_n}([L_n]) \in W$, there exists open neighborhoods W_j of $f_{i_j}([L_j])$ in G such that $W_1 W_2 \dots W_n \subseteq W$. That both compositions $f_i \circ \pi_i : \Omega(U_i) \rightarrow \pi_1^\tau(U_i) \rightarrow G$ are continuous means that, for each j , there is a basic open neighborhood $\mathcal{V}_j = \bigcap_{m=1}^{M_j} \langle K_{M_j}^m, A_m^j \rangle$ of L_j contained in $\pi_{i_j}^{-1}(f_{i_j}^{-1}(W_j)) \subseteq \Omega(U_{i_j})$. We may assume that M_j is divisible by 3 and that $A_m^j \subseteq U_1 \cap U_2$ whenever $L_j \left(K_{M_j}^m \right) \subseteq U_1 \cap U_2$. Since each p_j has image in $U_1 \cap U_2$, this assumption means that $A_m^j \subseteq U_1 \cap U_2$ whenever $K_{M_j}^m \subseteq [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$.

Taking restricted neighborhoods, we find that $(\mathcal{V}_j)_{[\frac{2}{3}, 1]}$ is an open neighborhood of p_j^{-1} for $j = 1, \dots, n-1$, $(\mathcal{V}_j)_{[\frac{1}{3}, \frac{2}{3}]}$ is an open neighborhood of α_j for $j = 1, \dots, n$, and $(\mathcal{V}_j)_{[0, \frac{1}{3}]}$ is an open neighborhood of p_{j-1} for $j = 2, \dots, n$. For $j = 1, \dots, n-1$ both $\left((\mathcal{V}_j)_{[\frac{2}{3}, 1]} \right)^{-1}$ and $(\mathcal{V}_{j+1})_{[0, \frac{1}{3}]}$ are neighborhoods of p_j so we may assume they are equal.

Since $p_j : I \rightarrow U_1 \cap U_2$ is well-targeted for each $j = 1, \dots, n-1$ with endpoint $p_j(1) = a_j$, there is an open neighborhood B_j of a_j in $U_1 \cap U_2$ such that for each $b \in B_j$, there is a path $\delta \in (\mathcal{V}_{j+1})_{[0, \frac{1}{3}]}$ from x_0 to b . Construct the neighborhood

$$\mathcal{U} = \bigcap_{j=1}^n \left((\mathcal{V}_j)_{[\frac{1}{3}, \frac{2}{3}]} \right)^{[t_{j-1}, t_j]} \cap \bigcap_{j=1}^{n-1} \langle \{t_j\}, B_j \rangle$$

of α in $\Omega(X, x_0)$. For any loop $\gamma \in \mathcal{U}$, notice that

- For each $j = 1, \dots, n$,

$$(\mathcal{V}_j)_{[\frac{1}{3}, \frac{2}{3}]} = \left(\left((\mathcal{V}_j)_{[\frac{1}{3}, \frac{2}{3}]} \right)^{[t_{j-1}, t_j]} \right)_{[t_{j-1}, t_j]}$$

is an open neighborhood of $\gamma_j = \gamma_{[t_{j-1}, t_j]}$ in $P(X)$.

- Since α_j has image in U_{i_j} , so does γ_j .
- If α_j has image in $U_1 \cap U_2$, then so does γ_j .
- For $j = 1, \dots, n-1$, since $\gamma(t_j) \in B_j$, there is a path

$$\delta_j \in (\mathcal{V}_{j+1})_{[0, \frac{1}{3}]} = \left((\mathcal{V}_j)_{[\frac{2}{3}, 1]} \right)^{-1} \subseteq \langle I, U_1 \cap U_2 \rangle$$

from x_0 to $\gamma(t_j)$.

Let $\delta_0 = \delta_n = c_{x_0}$ and define a loop β by demanding that $\beta_{[t_{j-1}, t_j]}$ is the loop $\delta_{j-1} * \gamma_{[t_{j-1}, t_j]} * \delta_j^{-1} \in \Omega(U_{i_j}, x_0)$ for $j = 1, \dots, n$. Note that $\beta_{[t_{j-1}, t_j]}$ has image in U_{i_j} and if α_j has image in $U_1 \cap U_2$, then so does $\beta_{[t_{j-1}, t_j]}$. Thus

$$\beta \simeq (\delta_0 * \gamma_{[t_0, t_1]} * \delta_1^{-1}) * \dots * (\delta_{j-1} * \gamma_{[t_{j-1}, t_j]} * \delta_j^{-1}) * (\delta_j * \gamma_{[t_j, t_{j+1}]} * \delta_{j+1}^{-1}) * \dots * (\delta_{n-1} * \gamma_{[t_{n-1}, t_n]} * \delta_n) \simeq \gamma$$

and $\Phi([\beta])$ is well-defined as the product

$$f_{i_1}([\beta_{[t_0, t_1]}])f_{i_2}([\beta_{[t_1, t_2]}]) \dots f_{i_n}([\beta_{[t_{n-1}, t_n]}])$$

in G . Moreover, for $j = 1, \dots, n$, we have $(\beta_{[t_{j-1}, t_j]})_C \in (\mathcal{V}_j)_C$ for $C = [0, \frac{1}{3}], [\frac{1}{3}, \frac{2}{3}], [\frac{2}{3}, 1]$. Therefore

$$\beta_{[t_{j-1}, t_j]} \in \bigcap_C \left((\mathcal{V}_j)_C \right)^C = \mathcal{V}_j \subseteq \pi_{i_j}^{-1}(f_{i_j}^{-1}(W_j)) \subseteq \Omega(U_{i_j}, x_0)$$

All together, we see that

$$\phi(\gamma) = \Phi([\gamma]) = \Phi([\beta]) = f_{i_1}([\beta_{[t_0, t_1]}]) f_{i_2}([\beta_{[t_1, t_2]}]) \dots f_{i_n}([\beta_{[t_{n-1}, t_n]}]) \in W_1 W_2 \dots W_n \subseteq W.$$

This completes the proof of the inclusion $\mathcal{U} \subseteq \phi^{-1}(W)$. Therefore ϕ is continuous. \square

Remark 4.24. While the condition that $U_1 \cap U_2$ be wep-connected is sufficient for the van Kampen theorem to hold, it is certainly not a necessary condition. For any path connected, non-wep-connected space X , the unreduced suspension SX is quotient of $X \times I$ by collapsing both $X \times \{0\}$ and $X \times \{1\}$ to points. Let U_1 and U_2 be the image of $X \times [0, \frac{2}{3})$ and $X \times (\frac{1}{3}, 1]$ in the quotient respectively. The open sets U_1, U_2 are contractible and the van Kampen theorem holds trivially even though $U_1 \cap U_2$ is not wep-connected.

Example 4.25. We can now compute $\pi_1^\tau(Y)$ from Example 4.12 by choosing an appropriate cover. For each $x \in X = \omega + 1$, let 0_x denote 0 in $e_x^1 = [-1, 1]$. Since $U_3 = Y - \bigcup_{x \in X} \{0_x\}$ is homotopy equivalent to $\Sigma(X_+)$, we have $\pi_1^\tau(U_3) \cong F_M(\omega + 1)$. The union $U_1 \cup U_3$ is X and $U_1 \cap U_3$ is wep-connected and 1-connected. Therefore, the van Kampen theorem applies and gives

$$\pi_1^\tau(Y) \cong \pi_1^\tau(U_1) * \pi_1^\tau(U_3) \cong F_M(\omega) * F_M(\omega + 1) \cong F_M(\omega \sqcup (\omega + 1)) \cong F_M(\omega + \omega)$$

where $\omega + \omega$ is the ordinal sum.

Example 4.26. The canonical isomorphism $F_M(A_1 \sqcup A_2) \cong F_M(A_1) * F_M(A_2)$ of topological groups is quickly recovered from the van Kampen theorem. Choose a space Y_i such that $\pi_0^{qtop}(Y_i) \cong A_i$. Let $X = \Sigma((Y_1 \sqcup Y_2)_+) \cong \Sigma((Y_1)_+) \vee \Sigma((Y_2)_+)$, V_i be the wep-connected, contractible neighborhood $Y_i \wedge [0, \frac{1}{3}) \sqcup (\frac{2}{3}, 1] \subset \Sigma((Y_i)_+)$, $U_1 = \Sigma((Y_1)_+) \vee V_2$, and $U_2 = V_1 \vee \Sigma((Y_2)_+)$. Since $U_1 \cap U_2 = V_1 \vee V_2$ is wep-connected and contractible the van Kampen theorem gives the middle isomorphism in:

$$F_M(A_1 \sqcup A_2) \cong \pi_1^\tau(\Sigma((Y_1 \sqcup Y_2)_+)) \cong \pi_1^\tau(\Sigma((Y_1)_+)) * \pi_1^\tau(\Sigma((Y_2)_+)) \cong F_M(A_1) * F_M(A_2).$$

The first and third isomorphisms come from Theorem 4.1 and the fact that $\pi_0^{qtop}(Y_1 \sqcup Y_2) = A_1 \sqcup A_2$. This observation illustrates why the notion of wep-connected intersection is an appropriate generalization of local path connectedness. Indeed, if we only consider the case where $U_1 \cap U_2$ is locally path connected, this application of the van Kampen theorem is restricted to discrete groups.

Corollary 4.27. *Given the hypothesis of Theorem 4.23, the homomorphism $F_M(\Omega(U_1)) * F_M(\Omega(U_2)) \rightarrow \pi_1^\tau(X)$ induced by the canonical maps $\Omega(U_i) \rightarrow \pi_1^\tau(X)$, $i = 1, 2$ is a topological quotient map.*

Proof. Since $F_M(\Omega(U_1)) * F_M(\Omega(U_2)) \cong F_M(\Omega(U_1) \sqcup \Omega(U_2))$ it suffices to show that $Q : F_M(\Omega(U_1) \sqcup \Omega(U_2)) \rightarrow \pi_1^\tau(X)$, $Q(\alpha_1 \dots \alpha_n) = [\alpha_1 * \dots * \alpha_n]$ is quotient. Let $\pi_i : \Omega(U_i) \rightarrow \pi_1^{qtop}(U_i)$ be the quotient map identifying path components. Since F_M preserves quotients, $F_M(\pi_1 \sqcup \pi_2)$ is quotient. The map

$$k : F_M(\pi_1^{qtop}(U_1) \sqcup \pi_1^{qtop}(U_2)) \rightarrow \pi_1^\tau(U_1) * \pi_1^\tau(U_2)$$

of Proposition 3.10 is also quotient. Additionally, the projection $k' : \pi_1^\tau(U_1) * \pi_1^\tau(U_2) \rightarrow \pi_1^\tau(U_1) *_{\pi_1^\tau(U_1 \cap U_2)} \pi_1^\tau(U_2)$ is quotient. Let $h : \pi_1^\tau(U_1) *_{\pi_1^\tau(U_1 \cap U_2)} \pi_1^\tau(U_2) \cong \pi_1^\tau(X)$ be the isomorphism of Theorem 4.23. The composite $Q = h \circ k' \circ k \circ F_M(\pi_1 \sqcup \pi_2)$ is quotient since it is the composition of quotient maps. \square

It is not true that the wedge of two wep-connected spaces is wep connected. For instance, let

$$CX = \frac{I \times X}{\{0\} \times X}$$

be the cone on $X = \{1, 2, \dots, \infty\}$ (the one-point compactification of the natural numbers) and have basepoint the image of $(1, \infty)$ in the quotient. Even though CX is wep-connected, $CX \vee CX$ is not wep-connected. Thus to apply the van Kampen theorem to a wedge of two spaces, we require the following lemma.

Lemma 4.28. *If (X_λ, x_λ) is a family of based, wep-connected spaces which are locally path connected at their basepoints and such that $\{x_\lambda\}$ is closed in X_λ for each λ , then $X = \bigvee_\lambda X_\lambda$ is wep-connected.*

Proof. Let x_0 be the canonical basepoint of X . Since X is locally path connected at x_0 , the constant path $c_{x_0} : I \rightarrow X$ is well-targeted. Let $z \in X_\lambda - \{x_\lambda\}$. It suffices to find a well-targeted path in X from x_0 to z . Since X_λ is wep-connected, there is a well-targeted path $\gamma : I \rightarrow X_\lambda$ from x_λ to z . Let $\mathcal{U} = \bigcap_{j=1}^n \langle K_n^j, U_j \rangle$ be an open neighborhood of $\gamma : I \rightarrow X_\lambda \hookrightarrow X$ in $P(X, x_0)$. Then $\mathcal{V} = \bigcap_{j=1}^n \langle K_n^j, U_j \cap X_\lambda \rangle$ is an open neighborhood of γ in $P(X_\lambda, x_0)$.

Since γ is well-targeted, there is an open neighborhood V of z in X_λ such that $x_\lambda \notin V$ and such that for each $v \in V$, there is a $\delta \in \mathcal{V}$ from x_λ to v . Note that V is an open neighborhood of z in X and for each $v \in V$ there is a path $\delta : I \rightarrow X_\lambda \hookrightarrow X$ in \mathcal{U} from x_0 to v . Thus $\delta : I \rightarrow X_\lambda \hookrightarrow X$ is well-targeted. \square

Theorem 4.29. *Let $(X, x_0), (Y, y_0)$ be path connected spaces having a countable neighborhood base of 1-connected neighborhoods at their basepoints. If there are wep-connected, simply connected neighborhoods A of x_0 in X and B of y_0 in Y , then there is a canonical isomorphism $\pi_1^\tau(X \vee Y) \cong \pi_1^\tau(X) * \pi_1^\tau(Y)$ of topological groups.*

Proof. We first recall a theorem of Griffiths [19]: If W_1, W_2 are based spaces, one of which has a countable base of 1-connected neighborhoods at its basepoint, then the inclusions $W_i \hookrightarrow W_1 \vee W_2$ induce an isomorphism $\pi_1(W_1) * \pi_1(W_2) \rightarrow \pi_1(W_1 \vee W_2)$ of groups. Since A, B , and $A \vee B$ are open in X, Y , and $X \vee Y$ respectively, each has a countable neighborhood base of path connected, 1-connected neighborhoods. Griffiths' theorem implies that $\pi_1(A \vee B) = 1$. Let $U_1 = X \vee B$ and $U_2 = A \vee Y$ so that $U_1 \cap U_2 = A \vee B$. Since A and B are wep-connected and locally path connected at their basepoints, $A \vee B$ is wep-connected by Lemma 4.28. The van Kampen theorem applies and gives an isomorphism $\pi_1^\tau(X \vee Y) \cong \pi_1^\tau(X \vee B) * \pi_1^\tau(A \vee Y)$ of topological groups. Again using Griffiths' theorem, the inclusions $X \hookrightarrow X \vee B$ and $Y \hookrightarrow A \vee Y$ induce continuous group isomorphisms $\pi_1^\tau(X) \rightarrow \pi_1^\tau(X \vee B)$ and $\pi_1^\tau(Y) \rightarrow \pi_1^\tau(A \vee Y)$. These group isomorphisms are also homeomorphisms since their inverses are induced by the retractions $X \vee B \rightarrow X$ and $A \vee Y \rightarrow Y$. All together, there are canonical isomorphisms

$$\pi_1^\tau(X \vee Y) \cong \pi_1^\tau(X \vee B) * \pi_1^\tau(A \vee Y) \cong \pi_1^\tau(X) * \pi_1^\tau(Y).$$

\square

5 Conclusions

Covering space theory, in the classical sense, provides general techniques for studying subgroups of given groups via topology. The three main results in Section 4 indicate that general topological groups, even those with complicated topological structure, are quite naturally realized as fundamental groups of simple wep-connected spaces (which are locally path connected at their basepoints). Thus it is plausible that there are techniques allowing one to study the structure of topological groups by studying the topology of spaces having non-trivial local properties.

The author has proposed a generalization of covering space theory [6] as one such technique to study open subgroup(oid)s of topologically enriched group(oid)s. This theory (of semicoverings) also indicates that the topology of $\pi_1^\tau(X)$ typically retains a good deal more information than the category of covering spaces of X . It is a great convenience that, unlike covering space theory, semicovering theory applies to arbitrary locally path connected spaces, the generalized wedges of circles of Section 4.1, and the CW-like spaces constructed in Section 4.2. It would be interesting if other properties such as separation and zero dimensionality of topological groups and their connection to shape injectivity could be studied using similar techniques.

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